# Geometry Processing

## 4 Parameterization

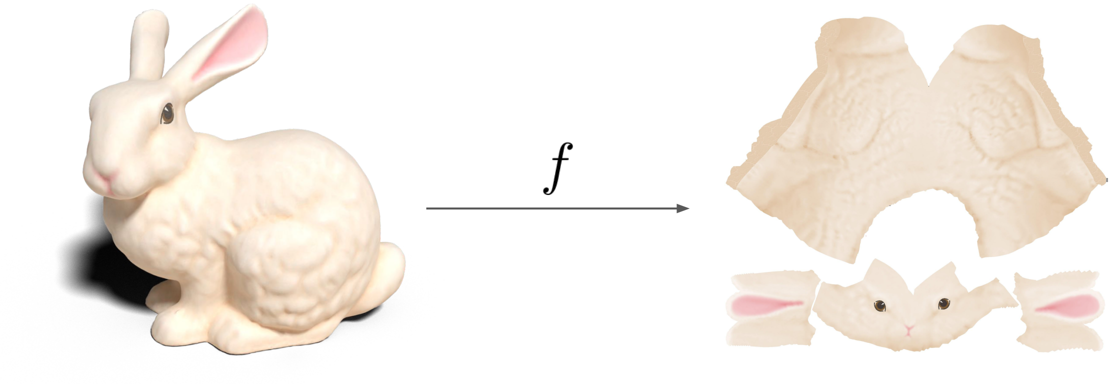
Ludwig-Maximilians-Universität München

# Session 4: Parameterization

- Motivation
- Methods
  - Tutte's Embedding Theorem (Barycentric Mapping)
  - Least Squares Conformal Maps (LSCM)
  - Angle-based Flattening (ABF)
- Summary
- Discussion: Parameterization in Blender

# Parameterization: Definition

A function $f$ that maps input surface in ***one-to-one*** correspondence with a different (e.g. 2D) domain



Example: In UV Mapping:

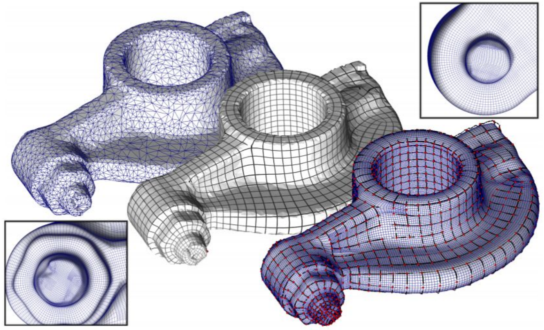
- Each vertex is associated with an UV coordinate $(x_i, y_i, z_i) \to (u_i, v_i)$
- Caution: vertices at seam

Equivalent terminologies: Flattening, unfolding

# Parameterization: Applications

Different types:

- Surface to plane mapping
    - Producing UV/normal/displacement/... maps
    - Compression **[Gu et al 2002] (UE5's Virtual Geometry)**
    - ...
- Plane to surface mapping
    - Remeshing (later)
    - ...
- Surface to surface mapping
    - Deformation (later)



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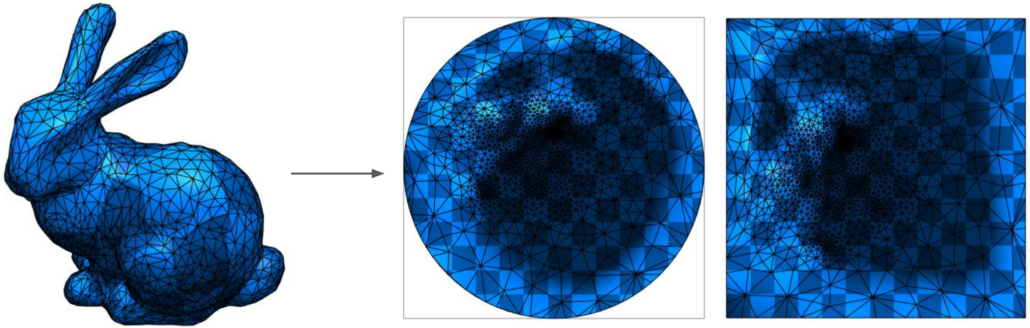
# Barycentric Mapping (Tutte's Embedding Theorem [Tutte 1960])

From graph theory: *Given a triangulated surface homeomorphic to a disk, if the (u, v) coordinates at the boundary vertices lie on a convex polygon in order, and if **the coordinates of the internal vertices are a convex combination of their neighbors**, then the (u, v) coordinates from a valid parameterization.*

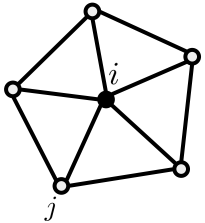


A convex polygon: circle, square, ...

Interior vertices: $\{1, ..., n_{\text{int}}\}$ Boundary vertices: $\{n_{\text{int}} + 1, ..., n\}$

A convex combination (*barycentric coordinates!*):

$$-w_{ii} \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \sum_{j \neq i} w_{ij} \begin{pmatrix} u_j \\ v_j \end{pmatrix}$$



# Barycentric Mapping: Matrix Form

For interior vertices:

$$-w_{ii}\begin{pmatrix}u_i\\v_i\end{pmatrix} = \sum_{j\neq i} w_{ij}\begin{pmatrix}u_i\\v_j\end{pmatrix} \Rightarrow \quad w_{ii} = -\sum_{i\neq j} w_{ij}$$

Recall Laplace matrix:

$$\mathbf{L} = \mathbf{DW} \qquad W_{ij} = \begin{cases} -\sum_{ik} w_{ik}, & \text{if } i = j \\ w_{ij}, & \text{if } j \text{ is a neighbor of } i \\ 0, & \text{otherwise} \end{cases}$$

$$\mathbf{W} = (W_{ij})$$

All we need is to solve two linear equations: $\mathbf{Lu}' = \mathbf{u} \quad \mathbf{Lv}' = \mathbf{v}$

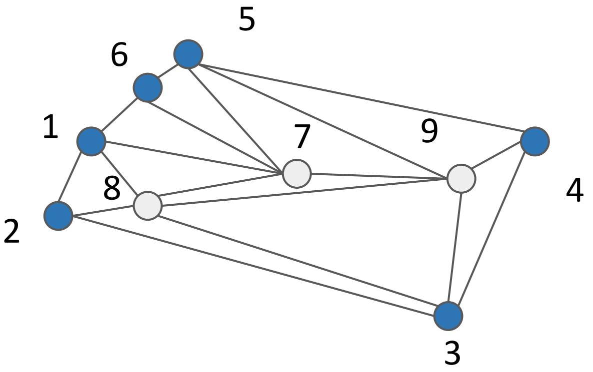
where the elements of $\mathbf{u}$ (respectively $\mathbf{v}$) is either zero (interior vertices) or precomputed (boundary vertices)

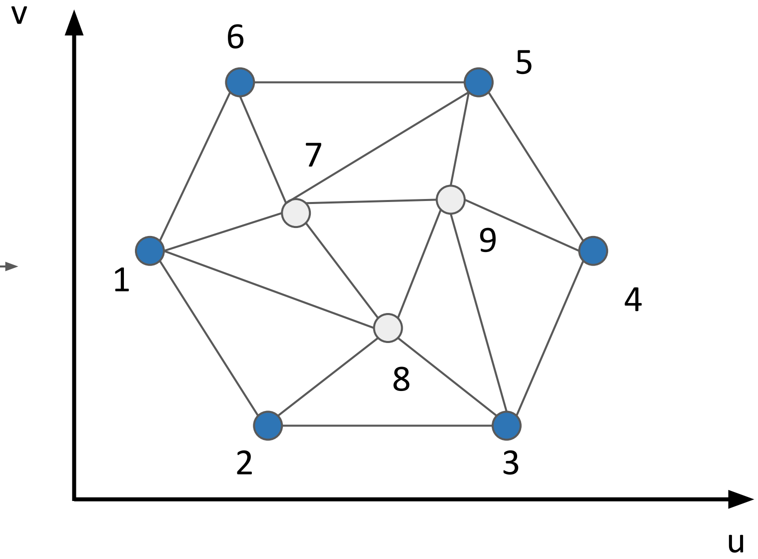
The solution $(\mathbf{u}', \mathbf{v}')$ is the barycentric mapped UV coordinates.

$$\Delta f = 0$$

**The Laplace Equation**

# Barycentric Mapping: Uniform Laplacian as Example



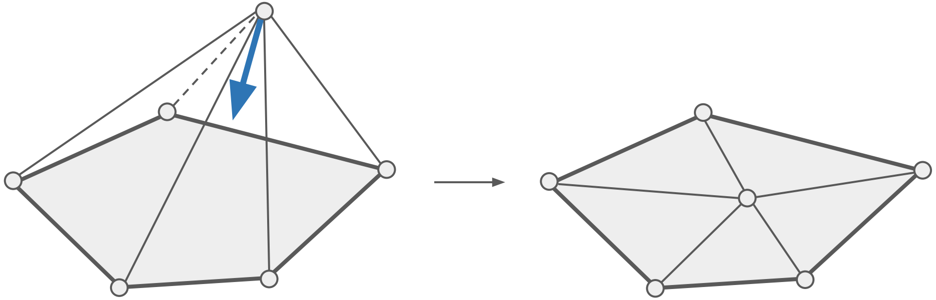

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & -5 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & -5 \end{pmatrix} \quad \mathbf{u}' = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} u_{v1} \\ u_{v2} \\ u_{v3} \\ u_{v4} \\ u_{v5} \\ u_{v6} \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow$$

$$\mathbf{L}\mathbf{u}' = \mathbf{u}$$

$$\mathbf{L}\mathbf{v}' = \mathbf{v}$$

# Barycentric Mapping: Intuition & Issues

Intuitively:

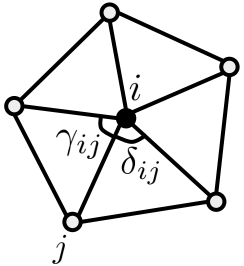


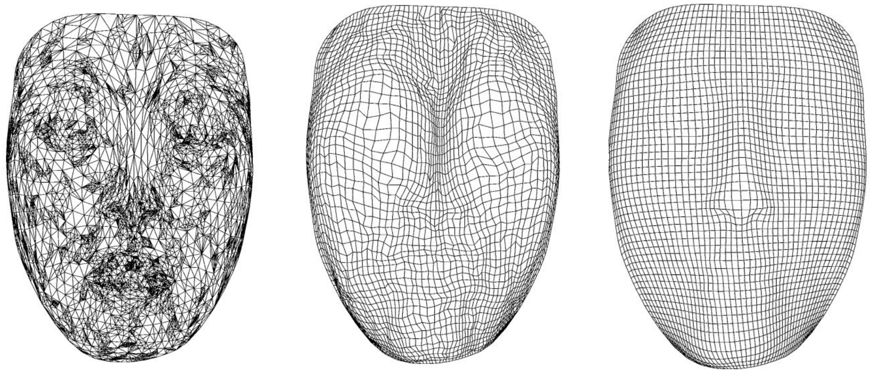
Different choice of Laplace matrix: Uniform, Cotan, ...

**Caution: Tutte's embedding theorem requires the Laplacian to satisfy: LOC+LIN+POS** (why?)

Cotan Laplacian can violates POS, there is a better version "*mean value weights*" **[Floater 03]** produce provably positive ones:

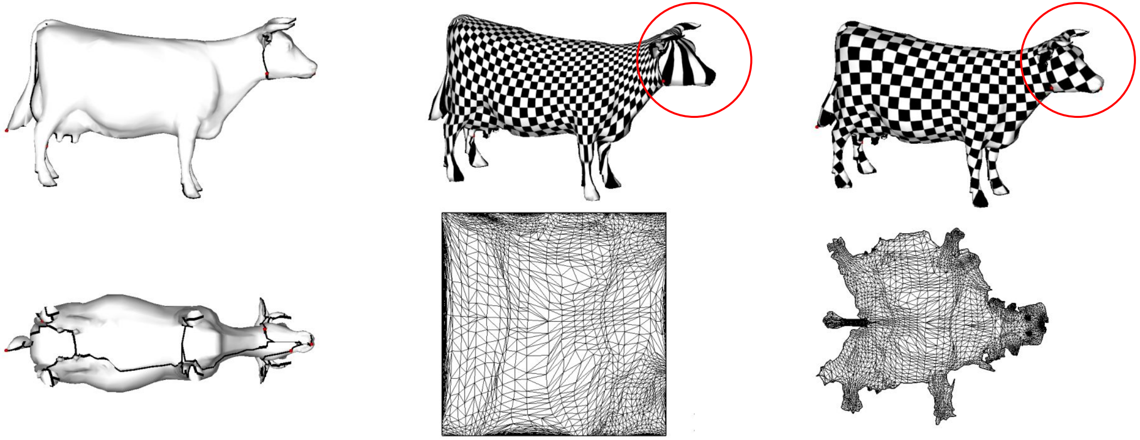
$$w_{ij} = \frac{1}{||f_i - f_j||}\left(\tan\left(\frac{\gamma_{ij}}{2}\right) + \tan\left(\frac{\delta_{ij}}{2}\right)\right)$$



# Barycentric Mapping: Issues (cont.)

Tutte's Embedding requires a fixed convex boundary ⇒ High distortion and mesh must have at least a boundary



Fixed Boundary

Free Boundary

How to minimize the distortion?

How to cut a "watertight" mesh?

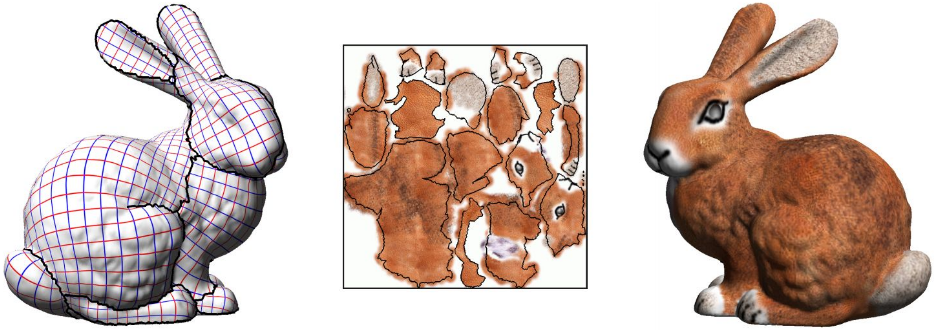
How to achieve free boundary?

# Texture Atlas Generation [Lévy et al 2002]

The generation of a texture atlas can be decomposed into the following steps:

1. **Segmentation**: The model is partitioned into a set of charts
2. **Parameterization**: Each chart is ‘unfolded’, i.e. put in correspondence with a subset of $\mathrm{R}^2$
3. **Packing**: The charts are gathered in texture space.

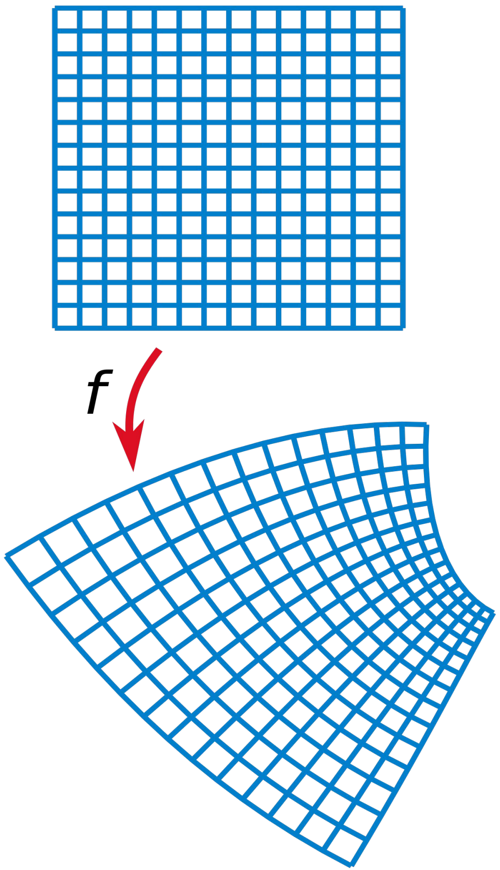
This workflow still largely exists in today's modeling practice (either manual mark seam or "smart" unwrap)



# Conformal Mapping

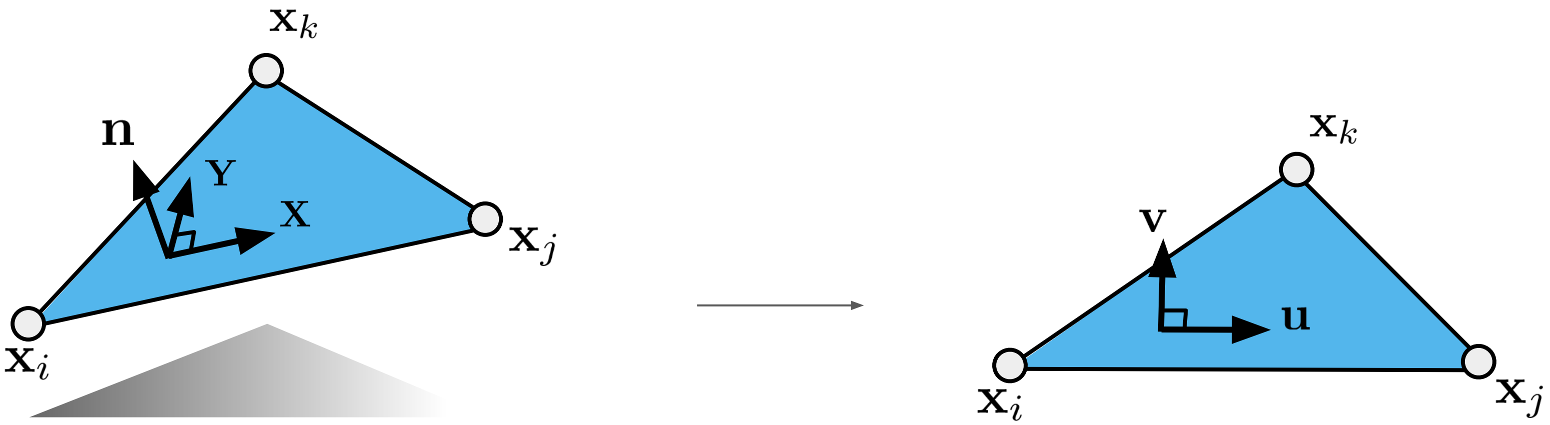
Conformal mappings = rotation + scale

Scale distortion is smoothly distributed (harmonic)




Equivalent terminologies: Angle-preserving, similar, scale

# Mapping in General




$$f(u+\Delta u, f+\Delta v) = f(u,v) + f_u(u,v)\Delta u + f_v(u,v)\Delta v \quad \Rightarrow f(u+\Delta u, v+\Delta v) = f(u,v) + \mathbf{J}\begin{pmatrix}\Delta u \\ \Delta v\end{pmatrix}$$

To achieve angle preservation, the Jacobian must be a similarity transformation:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = s\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \Rightarrow \quad \begin{array}{l} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{array}$$

(Surprisingly leads to *Cauchy-Riemann Equations*)

# Least Squares Conformal Maps (LSCM, ASAP) [Lévy et al 2002]

If the parameterization subject to $\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$ , then it is a conformal, but in general it is impossible (why?)

Instead, we minimize the least square "energy" as our objective:

$$E_{\mathrm{LSCM}} = \sum_t A_t \left( \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right)$$

(triangle area)

The energy measures non-conformality

It is invariant with respect to arbitrary translations and rotations

# Least Squares Conformal Maps (LSCM, ASAP) [Lévy et al 2002]

Issues:

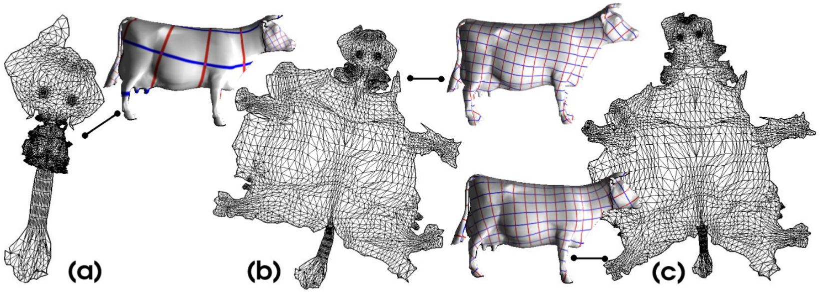
Energy does not have a unique minimizer, one can fix at least two vertices.

The choice of the vertices affects the results significantly

No guarantee on bijective

No guarantee on flip-free

...



# Angle-based Flattening (ABF)

More straightforward: Given angles for original mesh, find the closest angle that describe a flat mesh

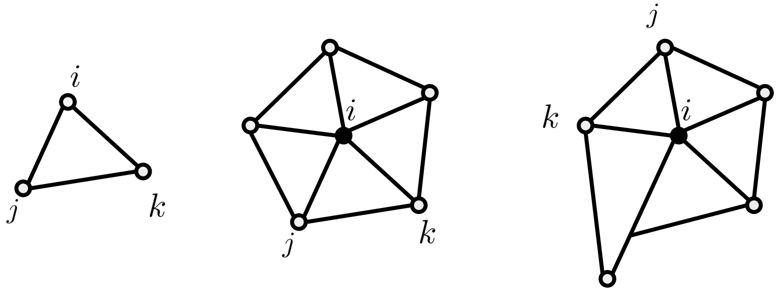
$$\min_{\theta} \sum_{i} (\hat{\theta}_i - \theta_i)^2$$

Subject to

1. angle sum: $\theta_i + \theta_j + \theta_k = \pi$

2. interior vertices sum: $\sum_{ijk} \theta_i = 2\pi$

3. compatible lengths around vertices (law of sines): $\prod_{ijk} \frac{\sin \theta_j}{\sin \theta_k} = 1$




Nonlinear optimization, many approximations: LinearABF **[Zayer et al 2007]**, ABF++ **[Sheffer et al 2005]**, etc.

You have the ability to known how they work exactly and be able to implement them eventually.

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# Summary

- Texture atlas generation workflow containing three parts: Segmentation, parameterization, and packing
- Tutte's embedding is a classic baseline for parameterization
- Objectives: **Minimize distortion (any kind of properties) with respect to a certain measure**
  - Validity (bijective, no self intersection): flipped triangle?
  - Boundary: fixed or free?
  - Domain: 2D or 3D?
  - ...
- A very large number of techniques exists that solves different problems (no best solution!)

# Further Readings


**[Tutte 1960]** (Tutte Embedding) Tutte WT. Convex representations of graphs. Proceedings of the London Mathematical Society. 1960.

**[Lévy et al 2002]** (LSCM) Lévy B, Petitjean S, Ray N, Maillot J. Least squares conformal maps for automatic texture atlas generation. ACM transactions on graphics (TOG). 2002.

**[Sheffer et al 2001]** (ABF) Sheffer A, de Sturler E. Parameterization of faceted surfaces for meshing using angle-based flattening. Engineering with computers. 2001.

**[Gu et al 2002]** Gu X, Gortler SJ, Hoppe H. Geometry images. InProceedings of the 29th annual conference on Computer graphics and interactive techniques 2002 Jul 1.

**[Floater 03]** Floater MS. Mean value coordinates. Computer aided geometric design. 2003 Mar.

**[Sheffer et al 2005]** (ABF++) Sheffer A, Lévy B, Mogilnitsky M, Bogomyakov A. ABF++: fast and robust angle based flattening. ACM Transactions on Graphics (TOG). 2005.

**[Zayer et al 2007]** Zayer R, Lévy B, Seidel HP. Linear angle based parameterization. in Eurographics SGP, Jul 2007.

**[Hormann et al 2007]** Kai Hormann, Bruno Lévy, and Alla Sheffer. Mesh parameterization: theory and practice Video files associated with this course are available from the citation page. In ACM SIGGRAPH Courses. 2007.

**[Smith et al 2015]** Smith J, Schaefer S. Bijective parameterization with free boundaries. ACM Transactions on Graphics (TOG). 2015 Jul 27.

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# Mesh Parameterization in Blender

## User Manual

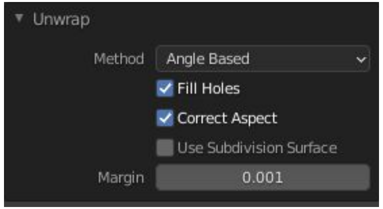
- https://docs.blender.org/manual/en/latest/modeling/meshes/uv/unwrapping/introduction.html
- https://docs.blender.org/manual/en/latest/modeling/meshes/editing/uv.html#unwrap
- https://docs.blender.org/manual/en/latest/modeling/meshes/editing/uv.html#smart-uv-project

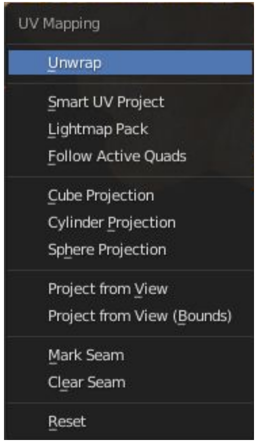
## API Manual

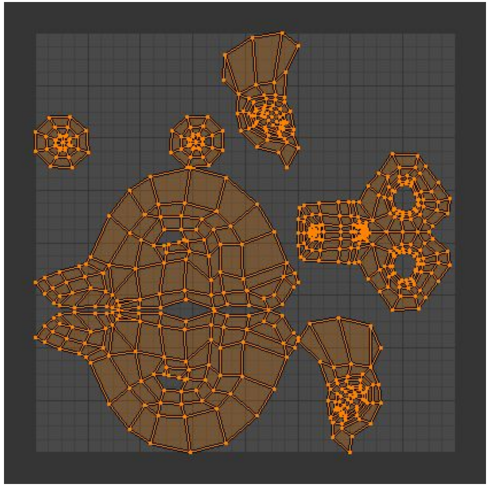
- https://docs.blender.org/api/current/bpy.ops.uv.html?highlight=uv#bpy.ops.uv.unwrap

## Implementation

- source/blender/editors/uvedit/uvedit_parameterizer.c (c2a01a6c118e)



# Blender: Geometry Nodes

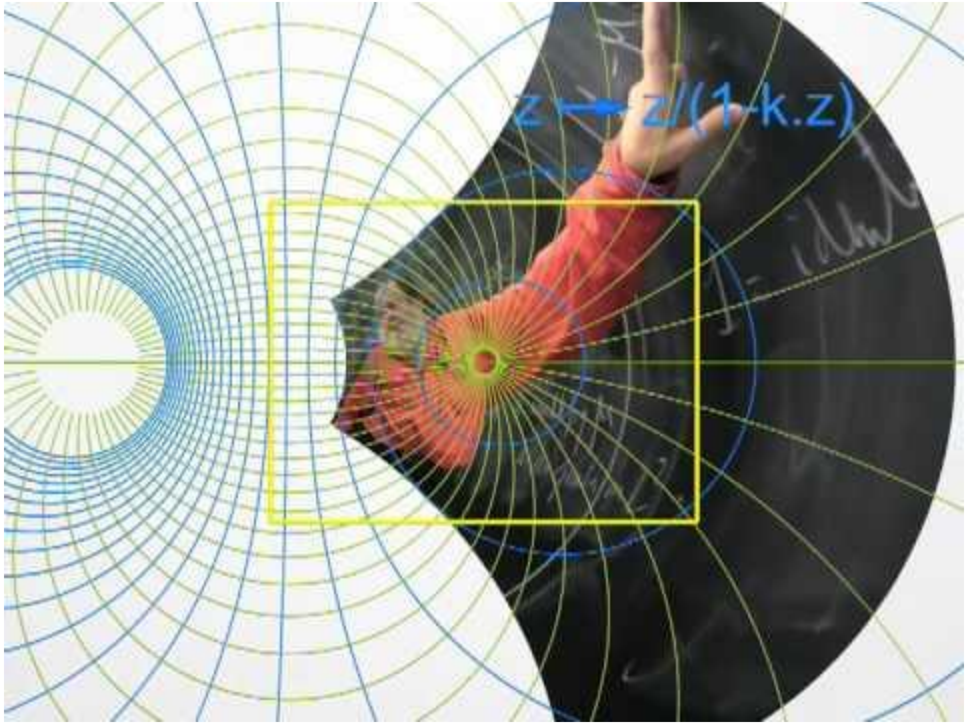
Beta release: https://builder.blender.org/download/

Manual: https://docs.blender.org/manual/en/dev/modeling/modifiers/nodes/index.html

Workboard: https://developer.blender.org/project/board/121/

# Dimensions: A walk through mathematics (2011)

https://www.youtube.com/watch?v=yJZP_-40KVw&list=PL97CCC2CC4E89C7E5




https://www.youtube.com/watch?v=MWHMzgZ4VIk