

Computergrafik 2: Filtern im Frequenzraum

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MHCI Lab, LMU München

Folien teilweise von Andreas Butz, sowie von Klaus D. Tönnies
(Grundlagen der Bildverarbeitung. Pearson Studium, 2005.)

Vorlesungen

Datum	Thema
24.4.	Einführung, Organisatorisches (Übungen, Klausur)
1.5./8.5.	keine Vorlesungen (wegen 1. Mai und CHI-Konferenz)
15.5.	Abtastung von Bildern, Punktbasierte Verfahren der Bildverbesserung
22.5.	Licht, Farbe, Farbmanagement
30.5.	Konvolution, Filterung im Ortsraum (Verschiebung wegen Pfingstdienstag)
5.6.	Fouriertransformation: Grundlagen
12.6.	Filterung im Frequenzraum
19.6.	Kanten, Linien, Ecken
26.6.	Segmentierung
3.7.	Segmentierung, Morphologische Operationen
10.7.	Klassifikation
17.7.	Image Matching
24.7.	Klausur (Hörsaal M 018 im Hauptgebäude, 14-16 Uhr)

Themen heute

- Korrelation im Frequenzraum
 - Finden von Templates
- Filtern im Frequenzraum
 - Bandreject/Bandpass-Filter
 - Notch-Filter, optimale Notch-Filter
- Dekonvolution
 - inverses Filtern
 - Wiener Filter

FILTERN IM FREQUENZRAUM

Frequenzraumfilterung

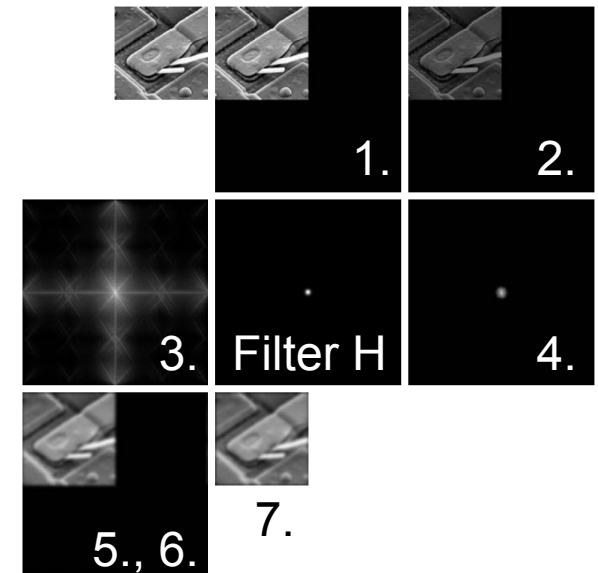
- Statt Konvolution im Ortsraum nun Multiplikation im Frequenzraum

$$f(x, y) * h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) \cdot h(x - m, y - n)$$

$$\Leftrightarrow F(u, v) \cdot H(u, v)$$

- Vorgehen

1. Padding im Ortsraum auf $2M, 2N$
2. Multiplikation mit $(-1)^{x+y}$
3. DFT
4. Multiplikation mit Filter im Frequenzraum
5. inverse DFT, Realteil
6. Multiplikation mit $(-1)^{x+y}$
7. Padding entfernen



Faltungstheorem

$H = \text{DFT}[h]$:

$$H(u) = \sum_{x=0}^{N-1} h(x) \cdot e^{-i2\pi(ux/N)}$$

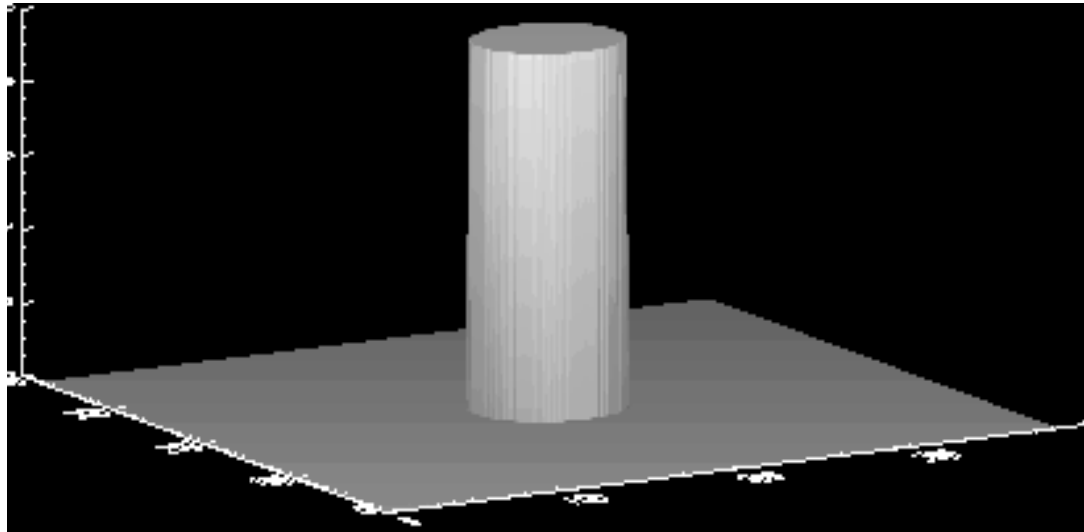
$h = \text{IDFT}[H]$:

$$h(x) = \frac{1}{N} \sum_{u=0}^{N-1} H(u) \cdot e^{i2\pi(ux/N)}$$

$$\begin{aligned} f(x) * h(x) &= \sum_{n=0}^{N-1} f(n) \cdot h(x-n) \\ &= \sum_{n=0}^{N-1} f(n) \cdot \frac{1}{N} \sum_{u=0}^{N-1} H(u) \cdot e^{i2\pi(u(x-n)/N)} \\ &= \frac{1}{N} \sum_{u=0}^{N-1} \left(\sum_{n=0}^{N-1} f(n) \cdot e^{-i2\pi(un/N)} \right) \cdot H(u) \cdot e^{i2\pi(ux/N)} \\ &= \frac{1}{N} \sum_{u=0}^{N-1} (F(u) \cdot H(u)) \cdot e^{i2\pi(ux/N)} \\ &= \text{IDFT}[F(u) \cdot H(u)] \end{aligned}$$

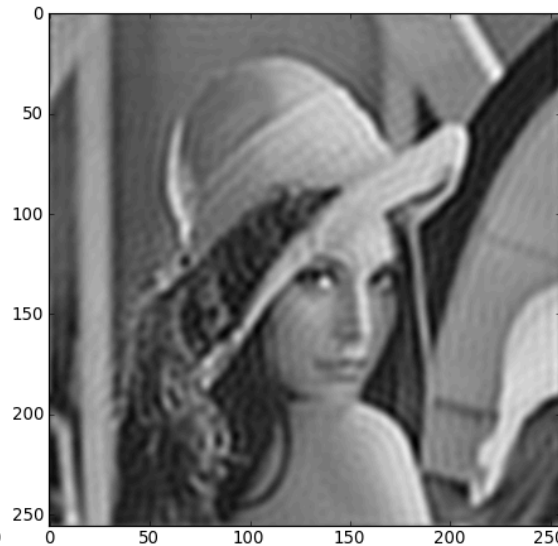
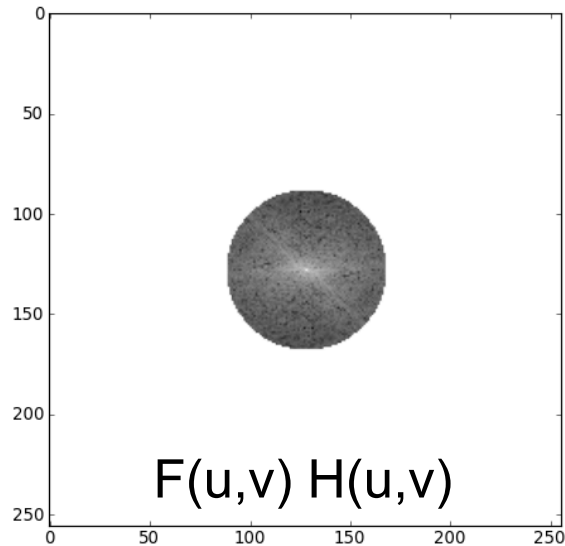
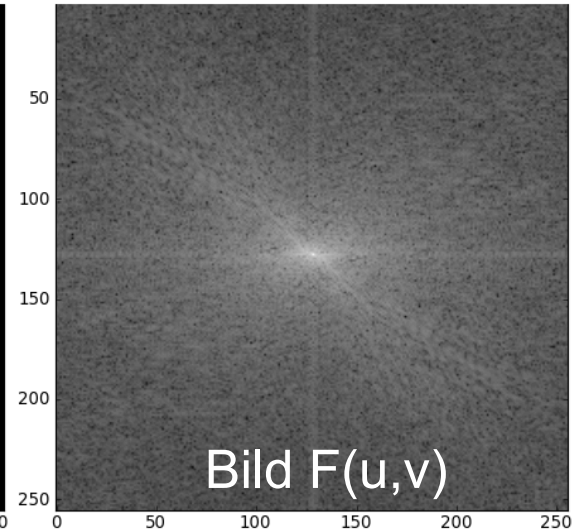
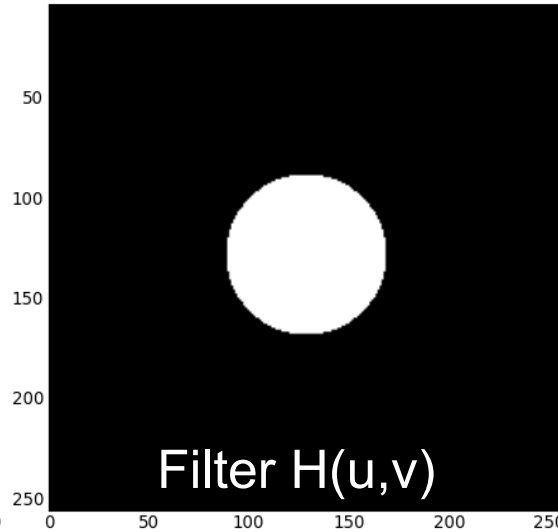
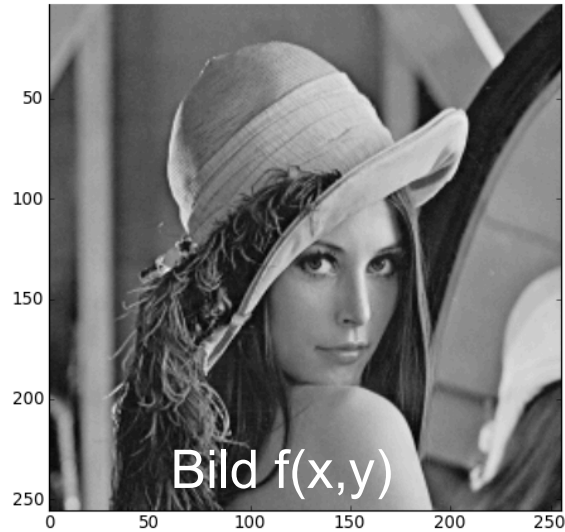
Ideales Tiefpassfilter

- Tiefpassfilter lässt tiefe Frequenzen passieren und dämpft hohe Frequenzen
- Ideales Tiefpassfilter $H_{F_{\max}}(u, v) = \begin{cases} 1 & , \text{ falls } u^2 + v^2 \leq F_{\max}^2 \\ 0 & , \text{ sonst.} \end{cases}$
 - ideal low pass filter (ILPF)
- F_{\max} : Cut-Off-Frequenz



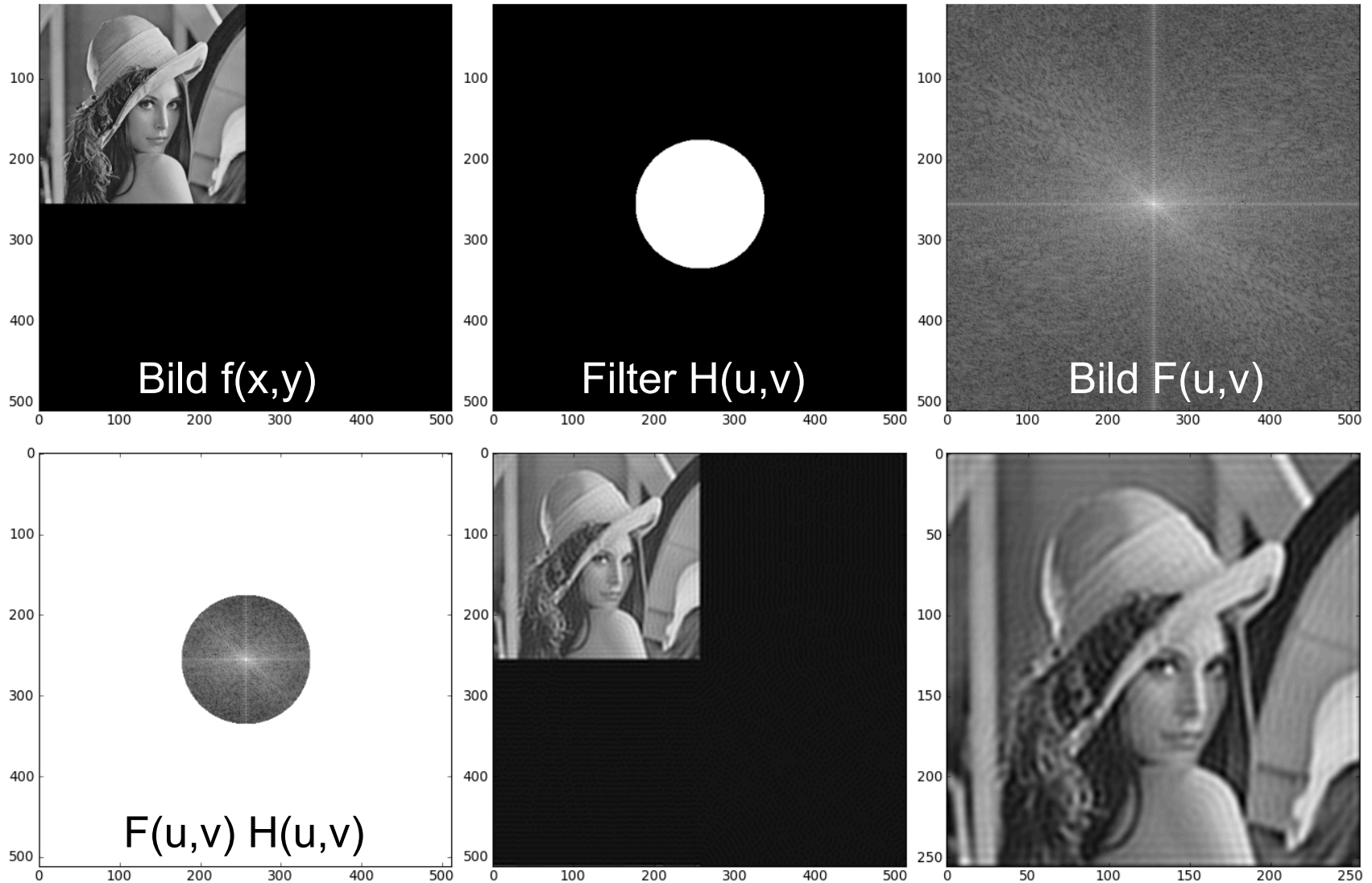
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Ideales Tiefpassfilter (ohne Padding)

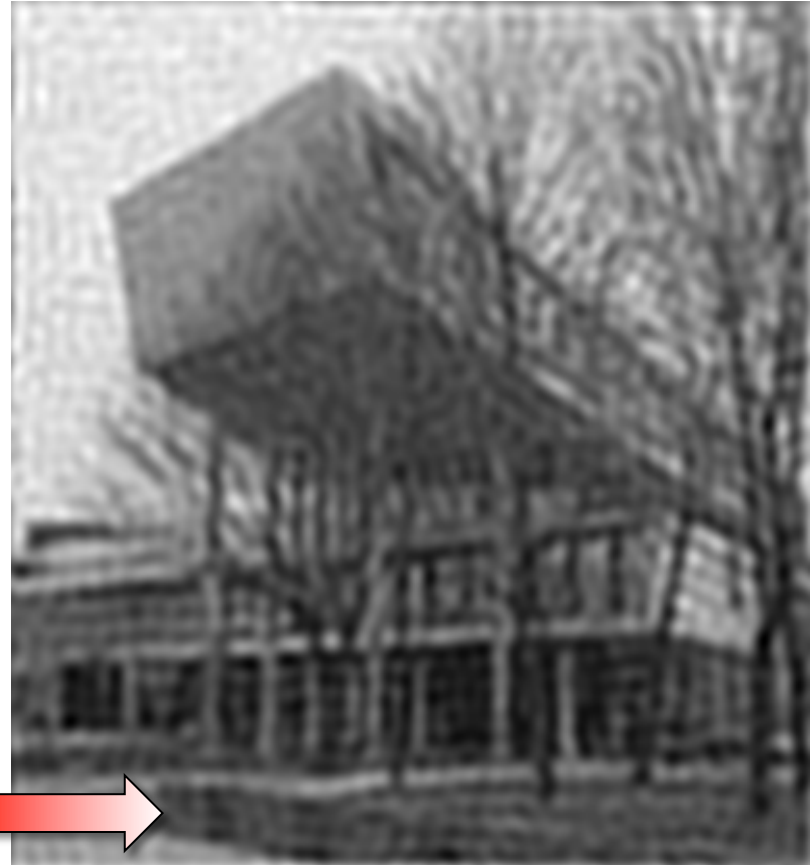
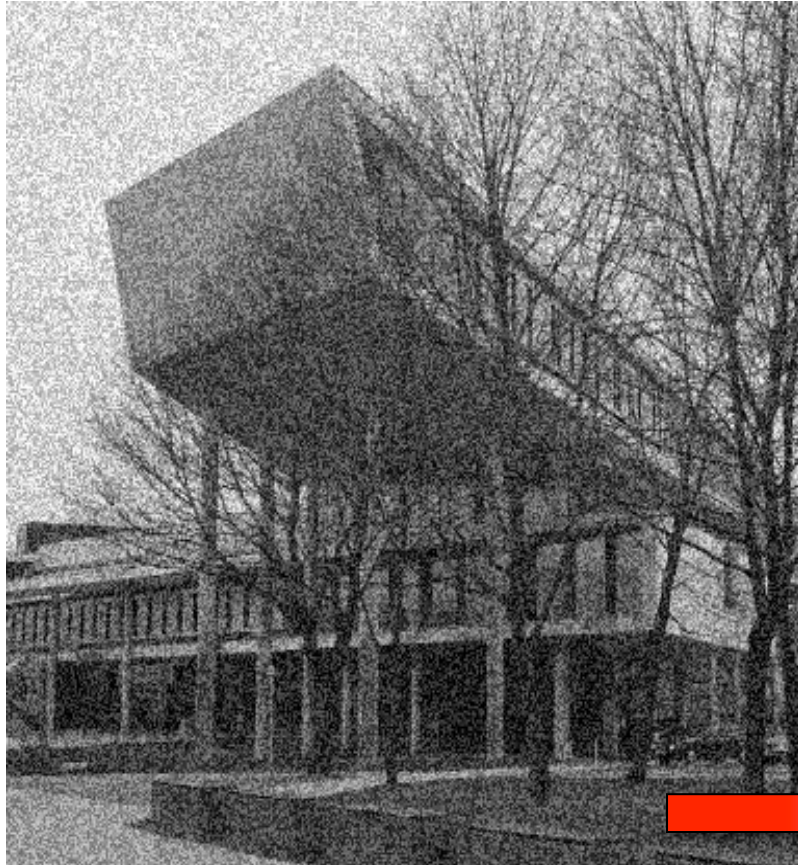


- Bild: 256x256 Pixel
- Filterradius (Cut-Off-Frequenz): 40 Pixel

Ideales Tiefpassfilter (mit Padding)



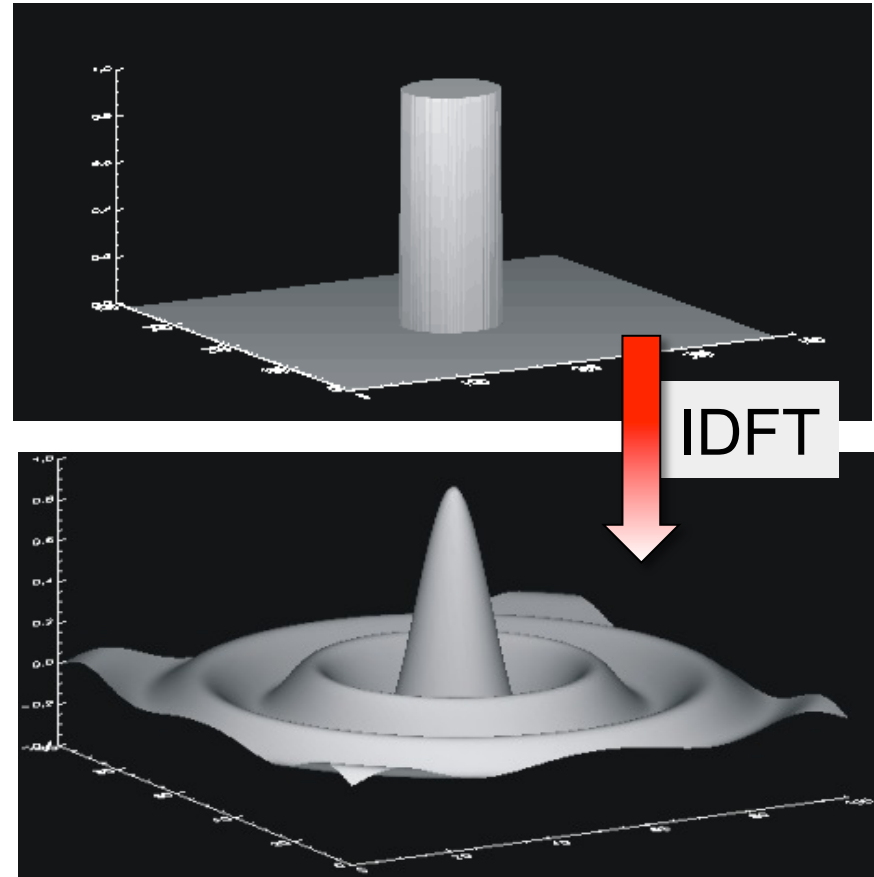
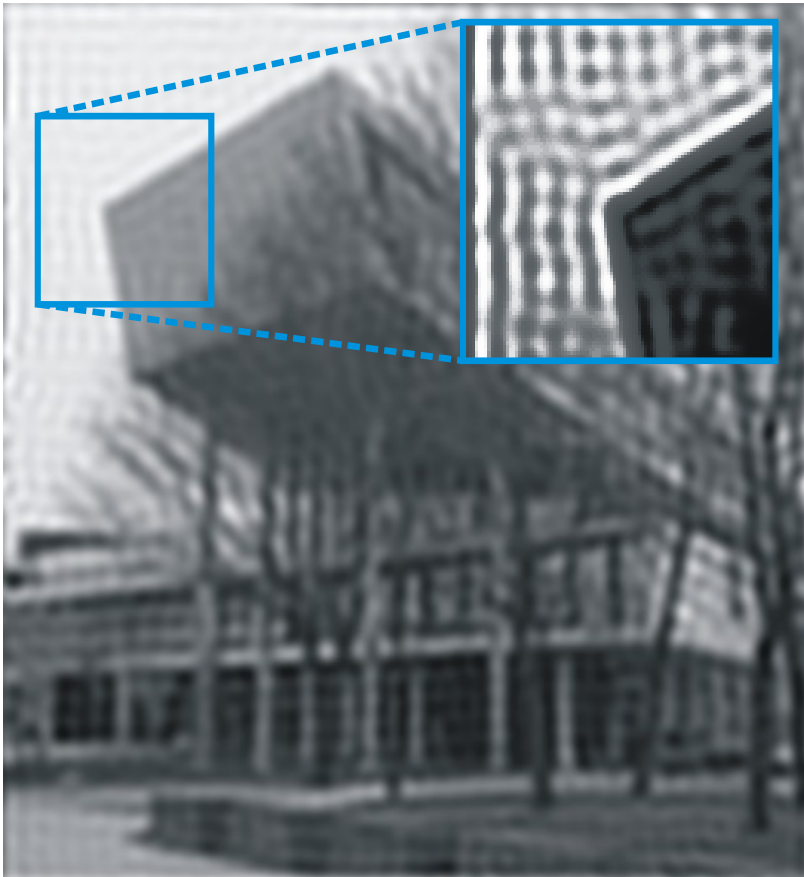
Ideales Tiefpassfilter zur Rauschunterdrückung



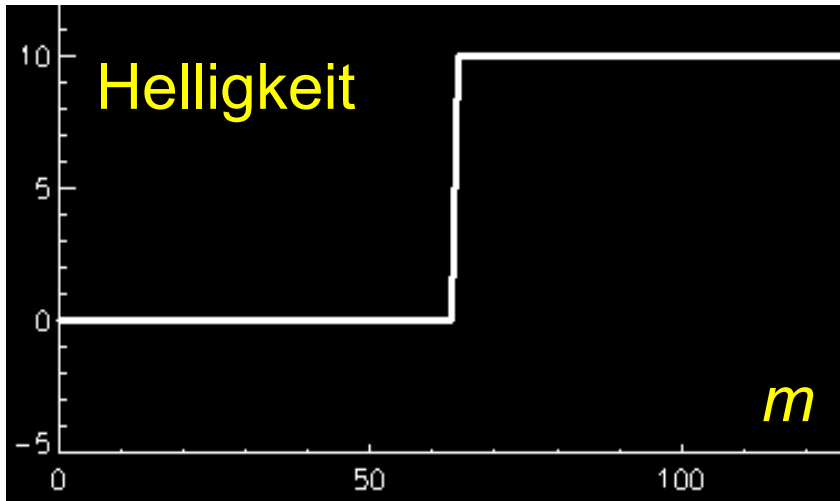
Cut-Off-Frequenz: 40

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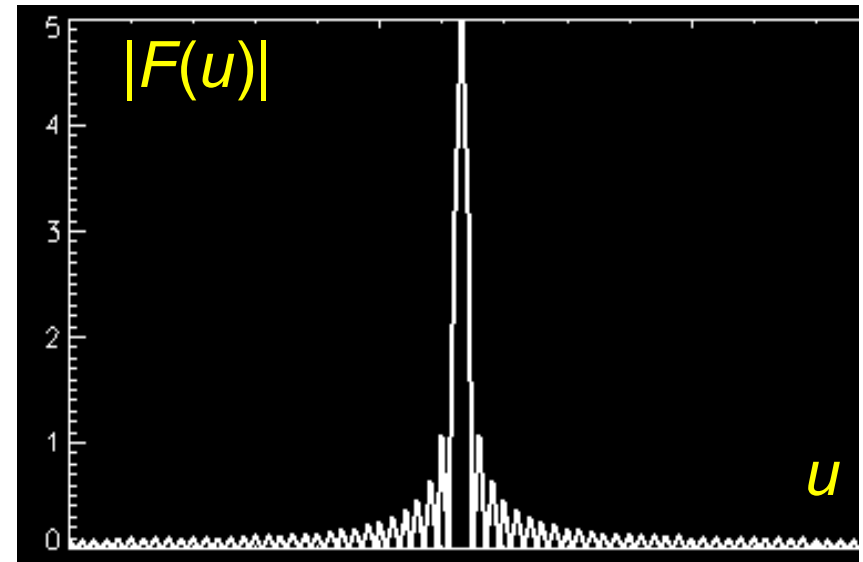
Ringing-Artefakt



Ringling-Artefakt



Bildzeile

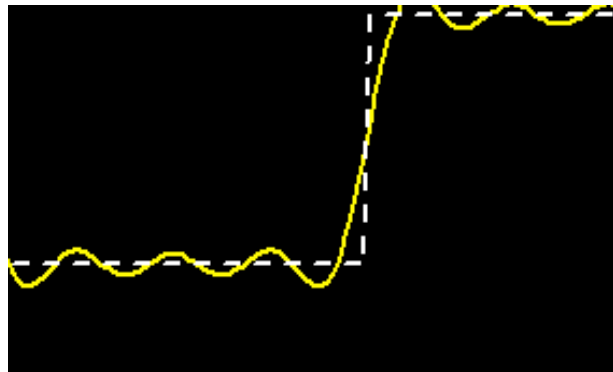
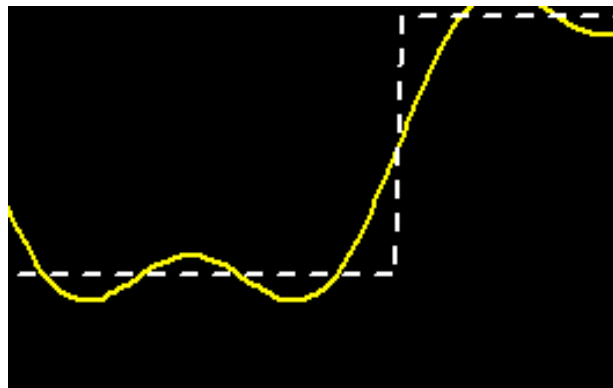


Fourier-transformierte Zeile

Das Ringing-Artefakt entsteht, weil scharfe Kanten durch Wellen **aller** Frequenzen beschrieben werden und der ideale LPF hohe Frequenzen abschneidet

Ringling-Artefakt

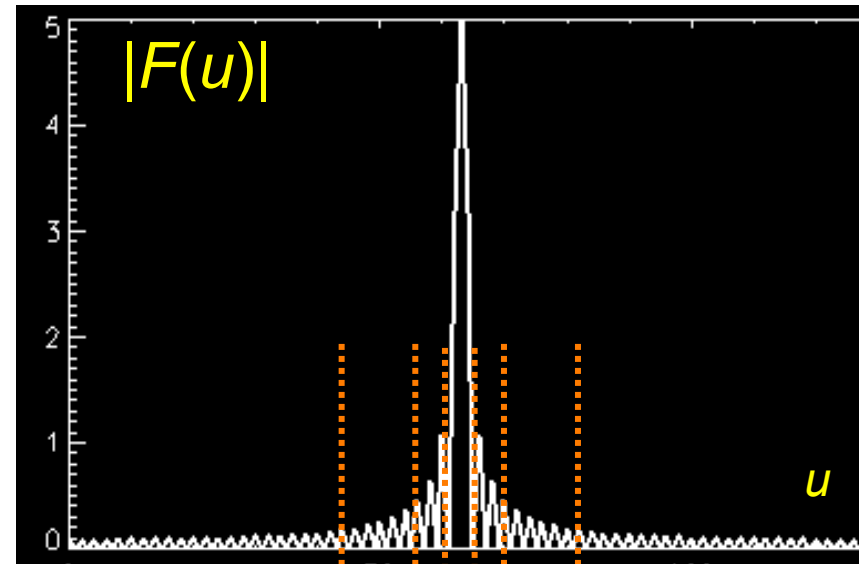
je kleiner die Cut-Off-Frequenz F_{\max} , desto stärker der Ringling-Artefakt



IDFT

IDFT

IDFT



invers Fourier-transformierte Zeile

Butterworth-Filter

- Frequenzen werden nicht gelöscht, sondern nur abgeschwächt

- Tiefpass-Filter (BLPF):

$$H(u, v) = \frac{1}{1 + (D(u, v) / D_0)^{2n}}$$

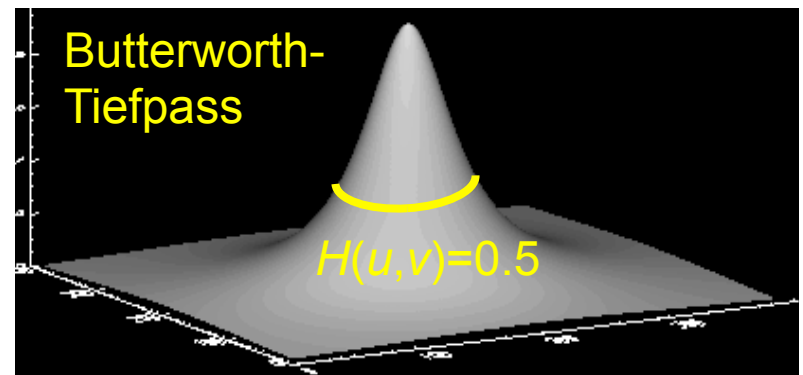
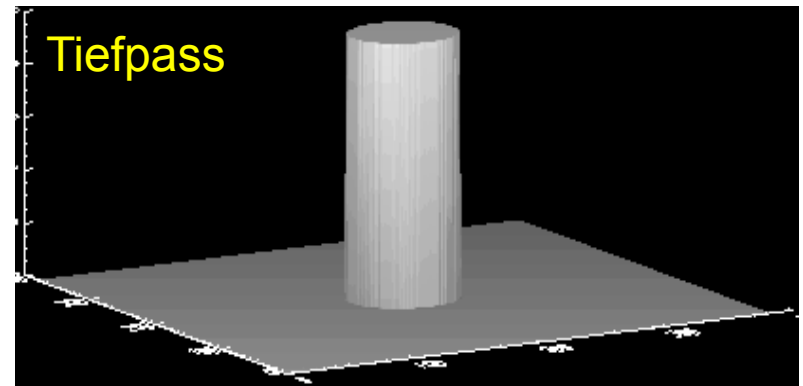
- Hochpass-Filter (BHPF):

$$H(u, v) = \frac{1}{1 + (D_0 / D(u, v))^{2n}}$$

D_0 : Cutoff-Frequenz

$D(u, v)$: Frequenz, Abst. Ursprung

n : Ordnung des Filters

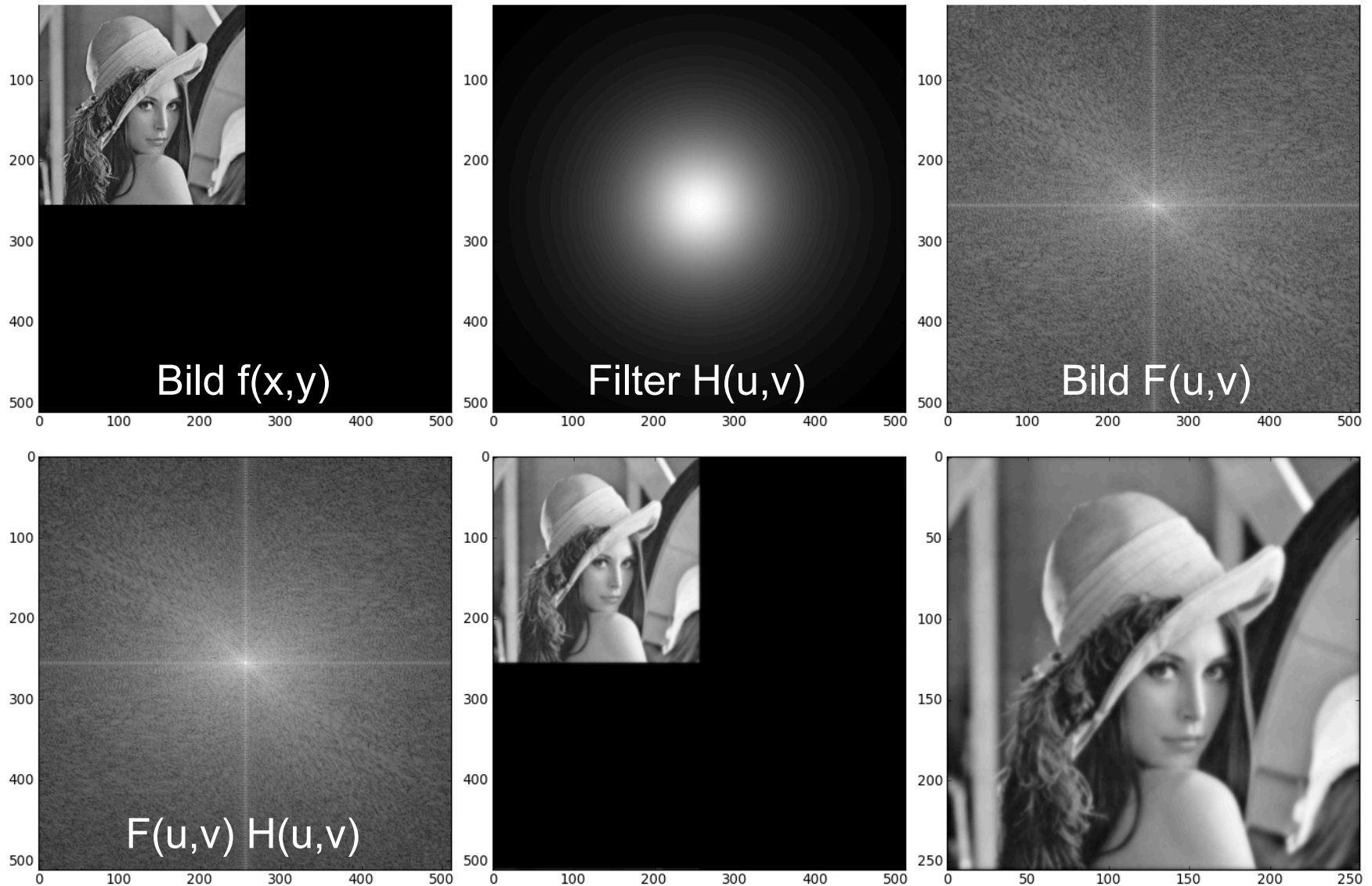


Einfacher Tiefpass vs. Butterworth

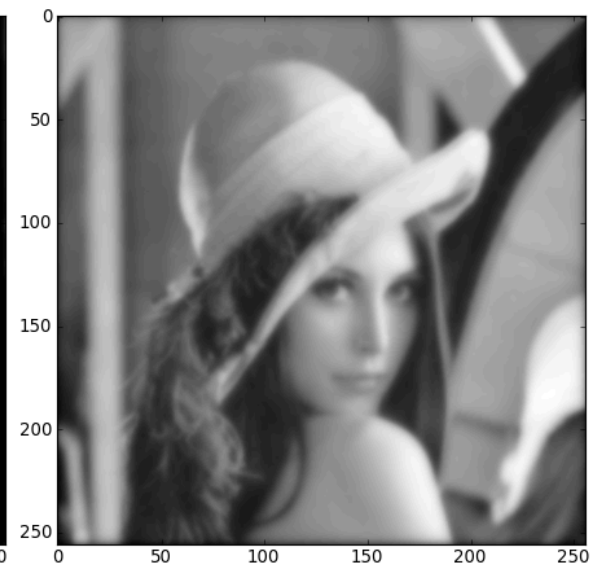
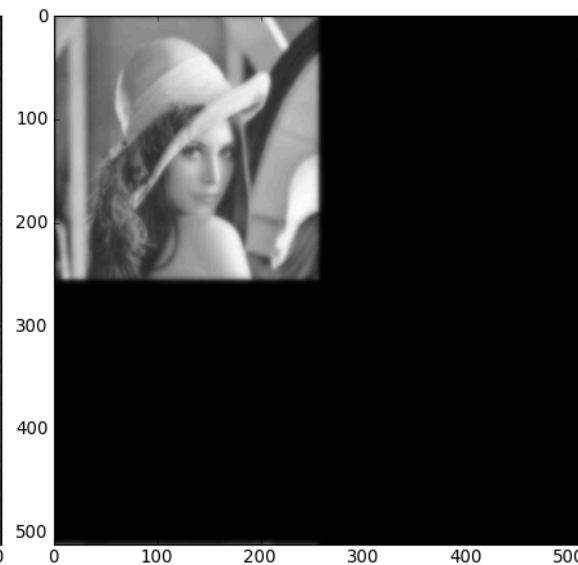
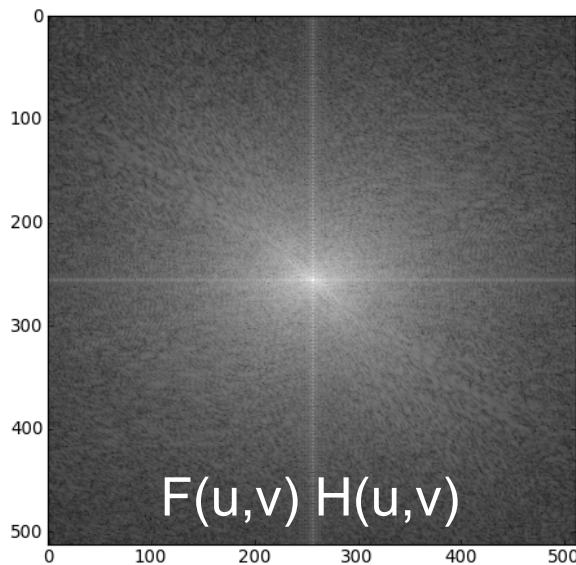
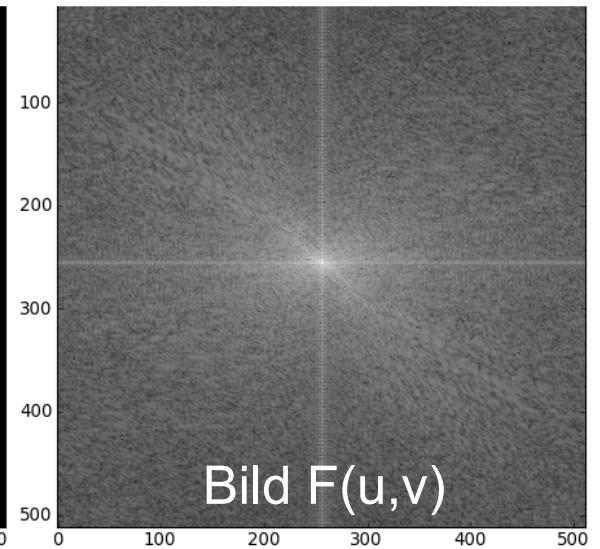
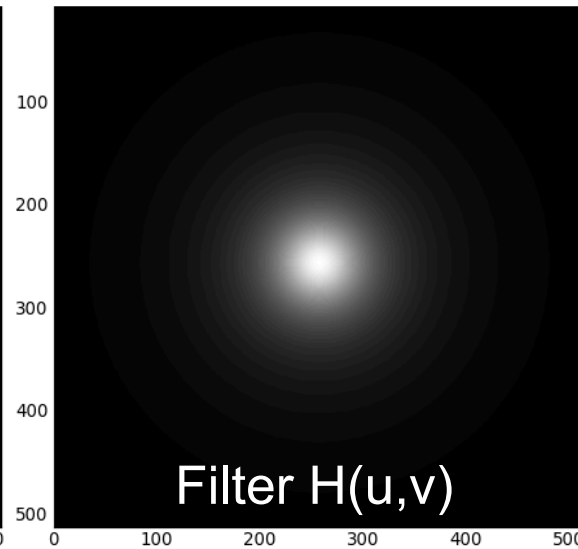
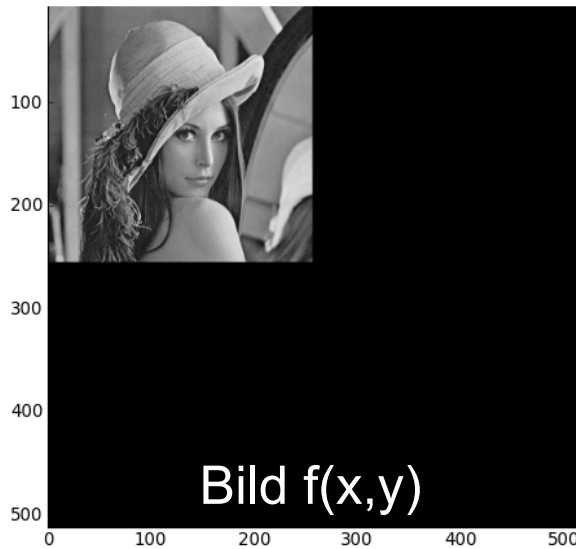


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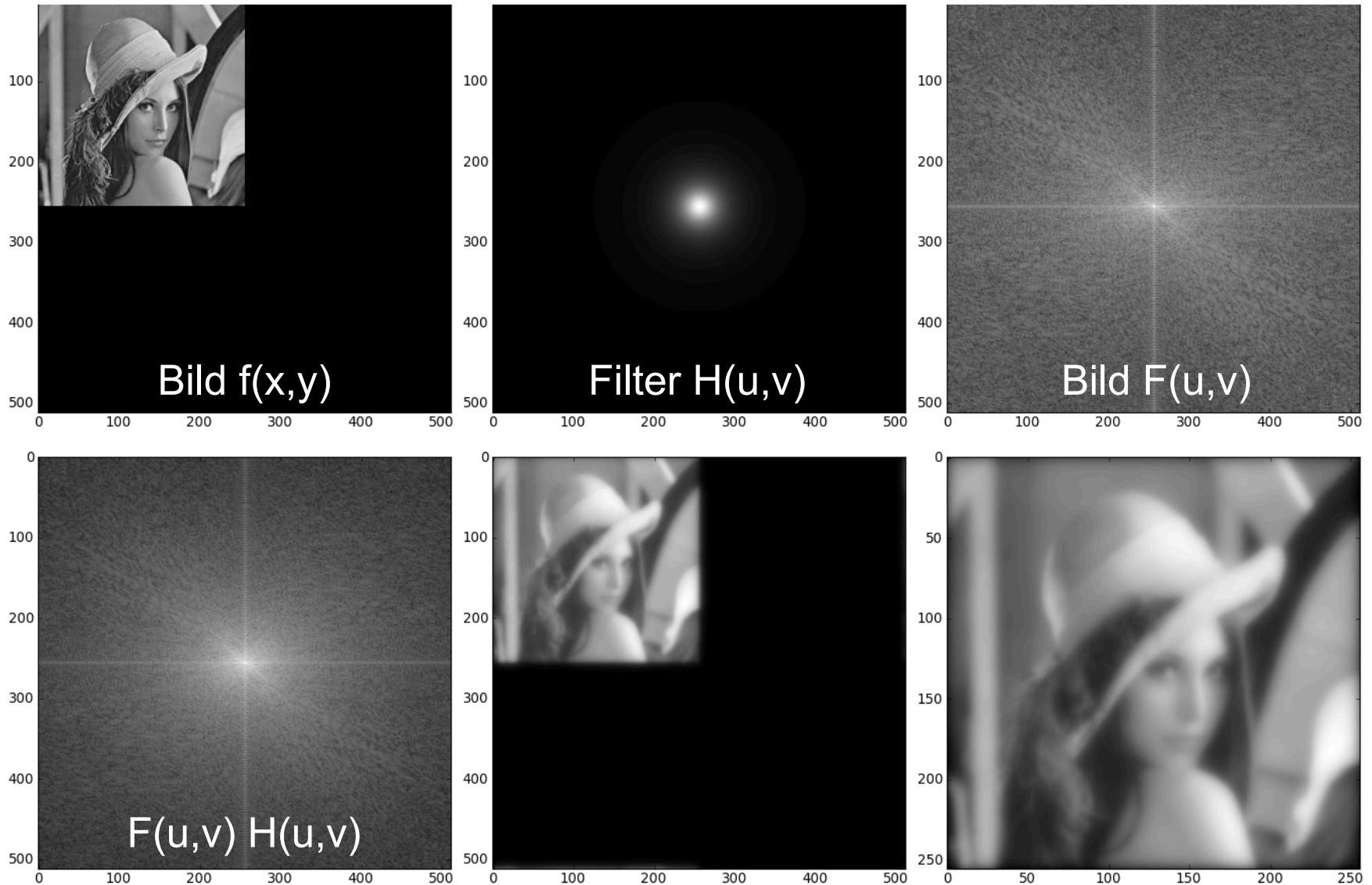
Beispiel: BLPF, $n = 2$, $D_0 = 80$



Beispiel: BLPF, $n = 2$, $D_0 = 40$

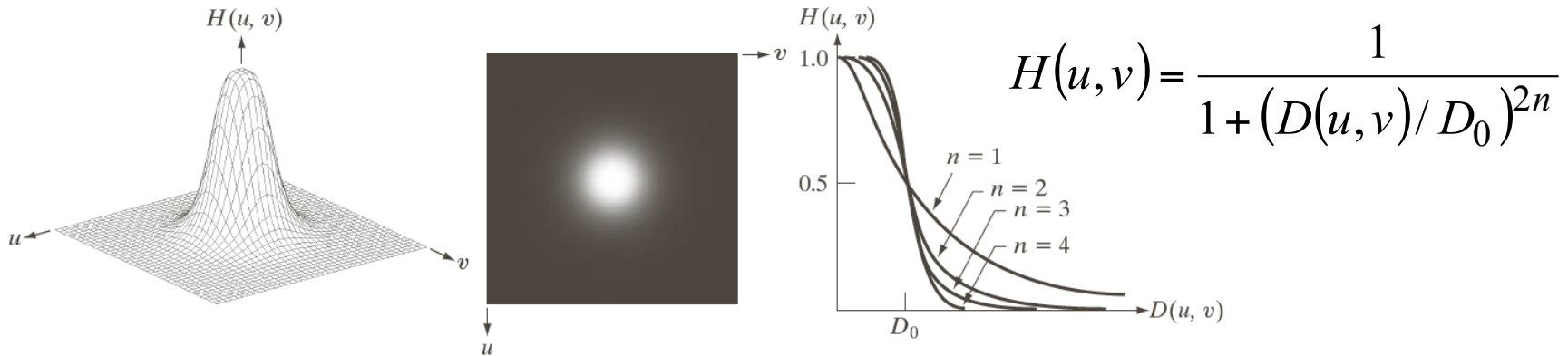


Beispiel: BLPF, $n = 2$, $D_0 = 20$



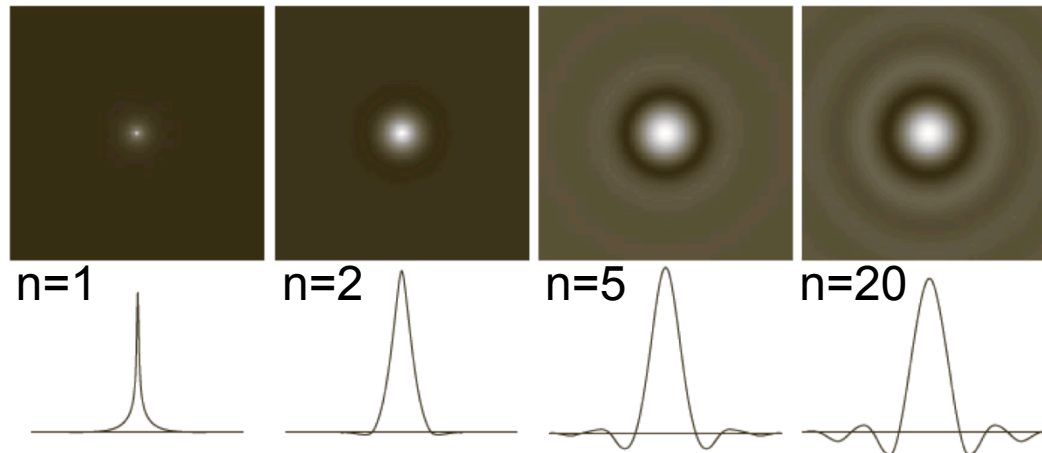
Ordnung n des Butterworth-Filters

- Ordnung n bestimmt die Steilheit der Transferfunktion



- je größer Ordnung n, desto stärkere Ringing-Artefakte

$M=N=1000$
 $D_0=5$



© R. C. Gonzalez & R. E. Woods, Digital Image Processing

Gauß-Filter

- Keine Artefakte, da Fourier-Transformation einer Gauß-Funktion wieder eine Gauß-Funktion

- Tiefpass-Filter (GLPF):

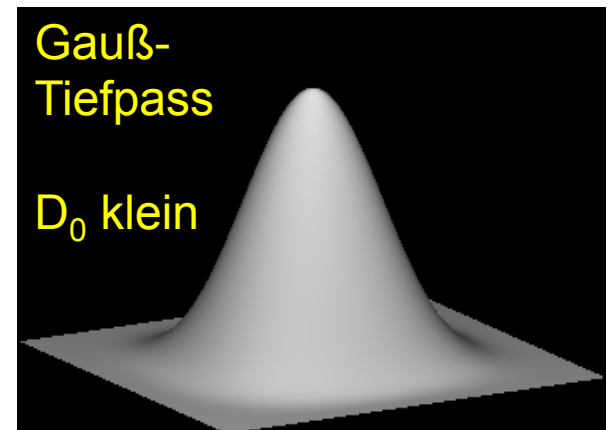
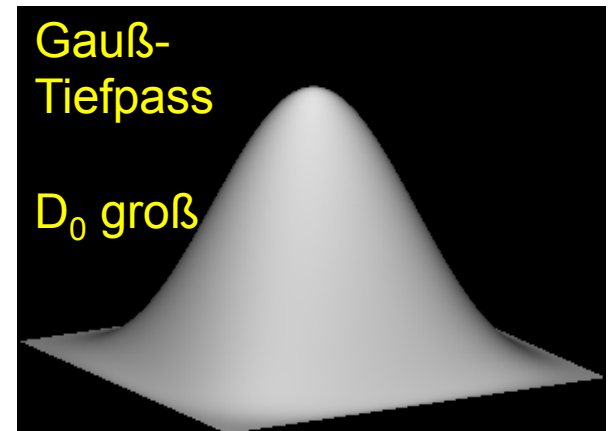
$$H(u, v) = \exp\left(-\frac{D^2(u, v)}{2D_0^2}\right)$$

- Hochpass-Filter (GHPF):

$$H(u, v) = 1 - \exp\left(-\frac{D^2(u, v)}{2D_0^2}\right)$$

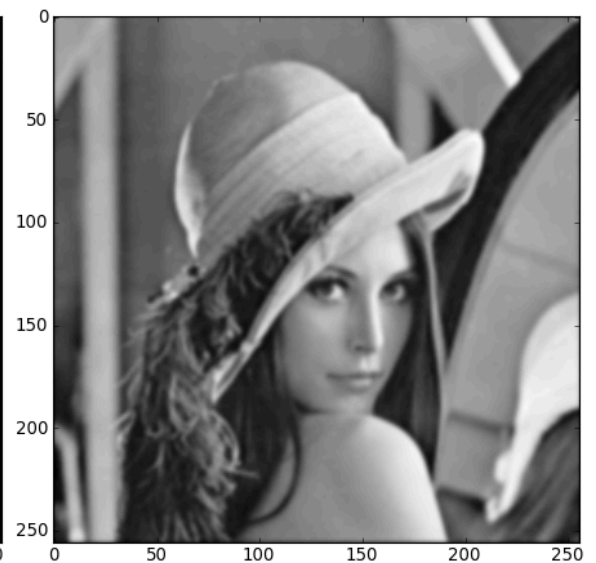
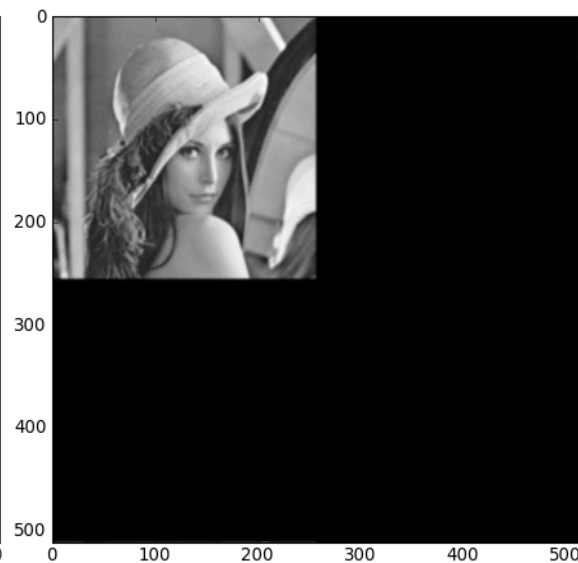
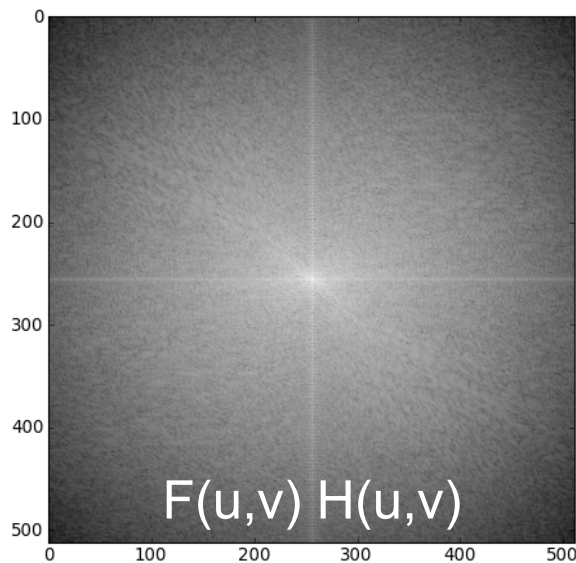
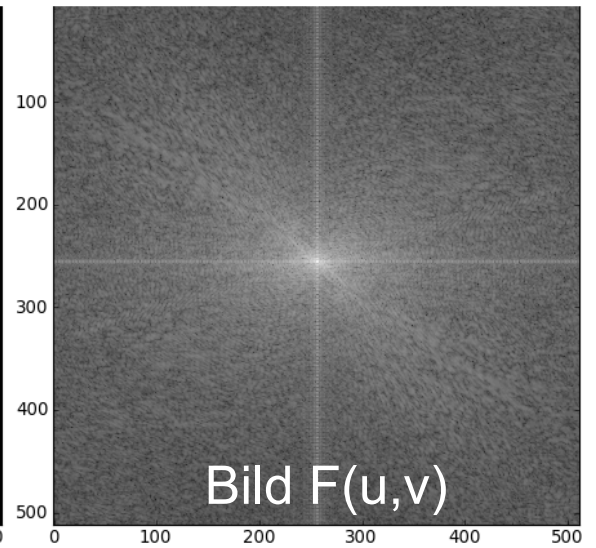
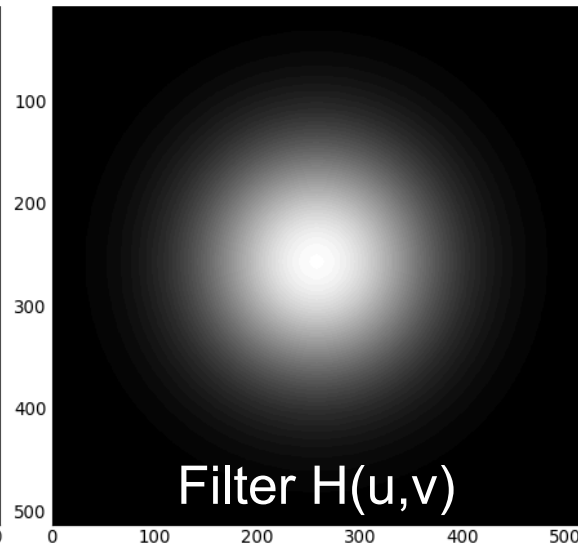
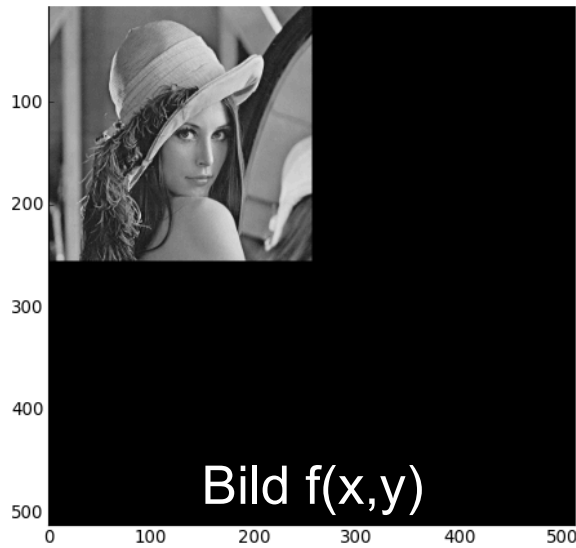
D_0 : entspricht σ

$D(u, v)$: Frequenz, Abstand vom Ursprung

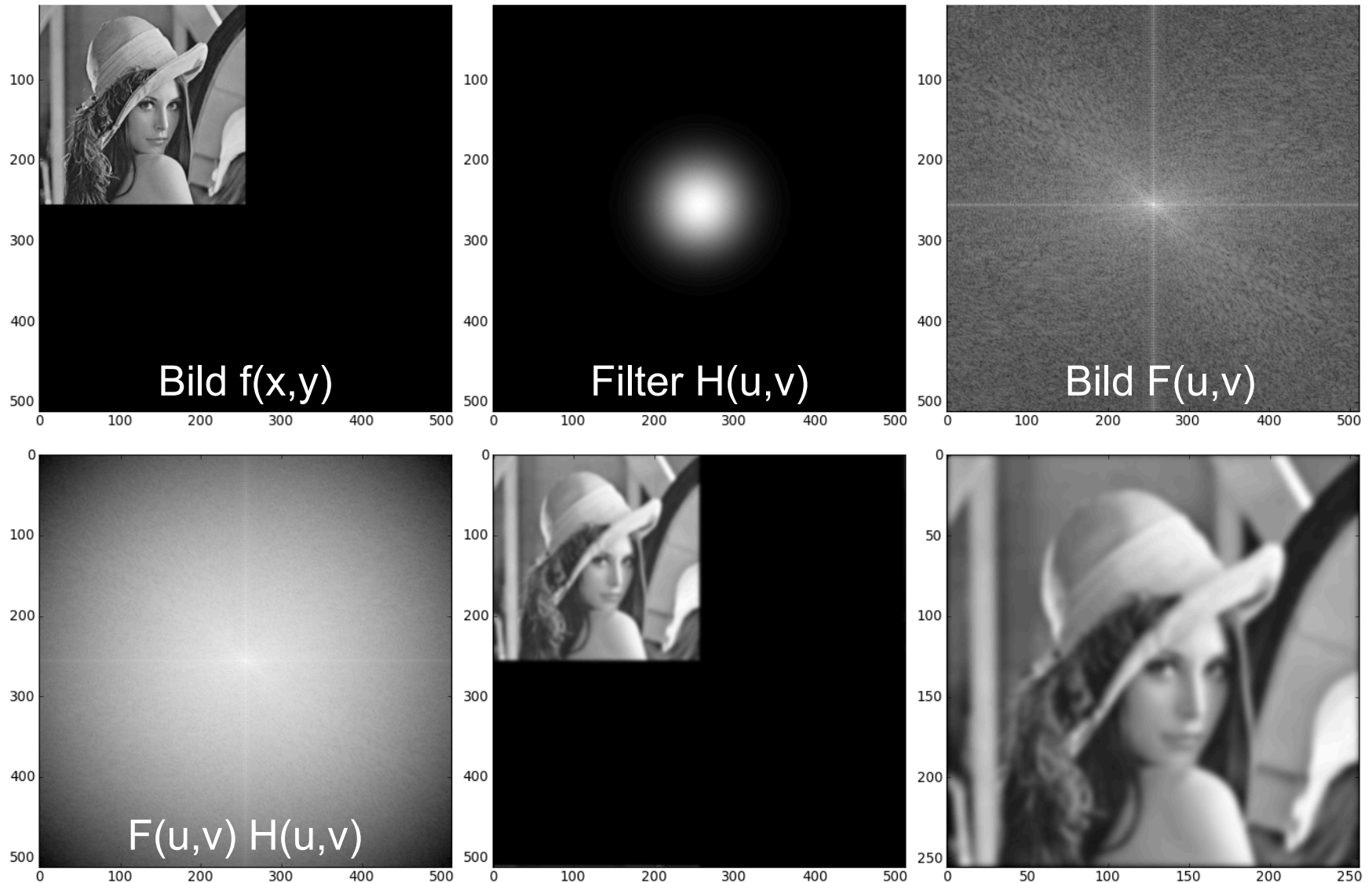


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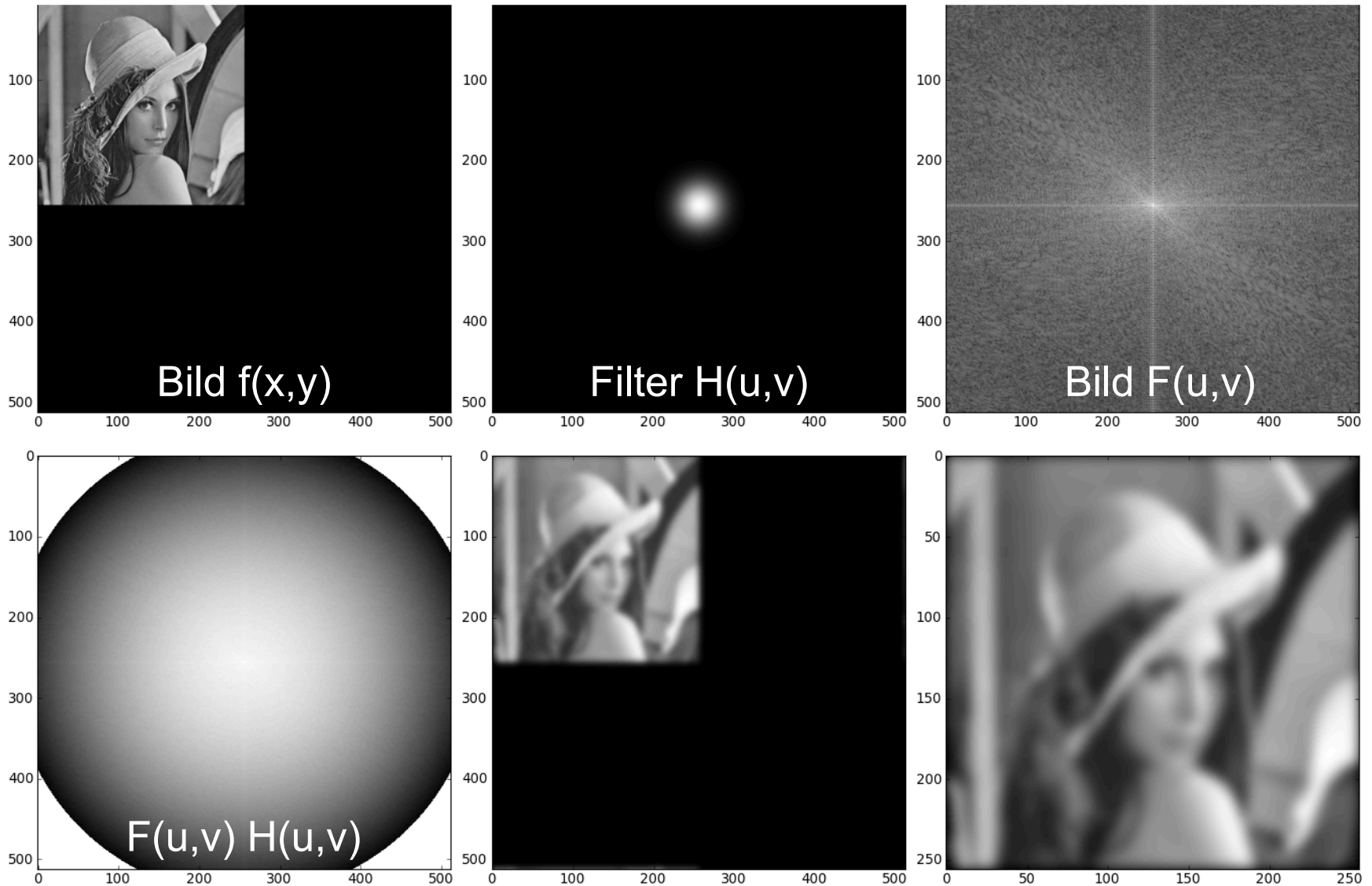
Beispiel: GLPF, $D_0 = 80$



Beispiel: GLPF, $D_0 = 40$

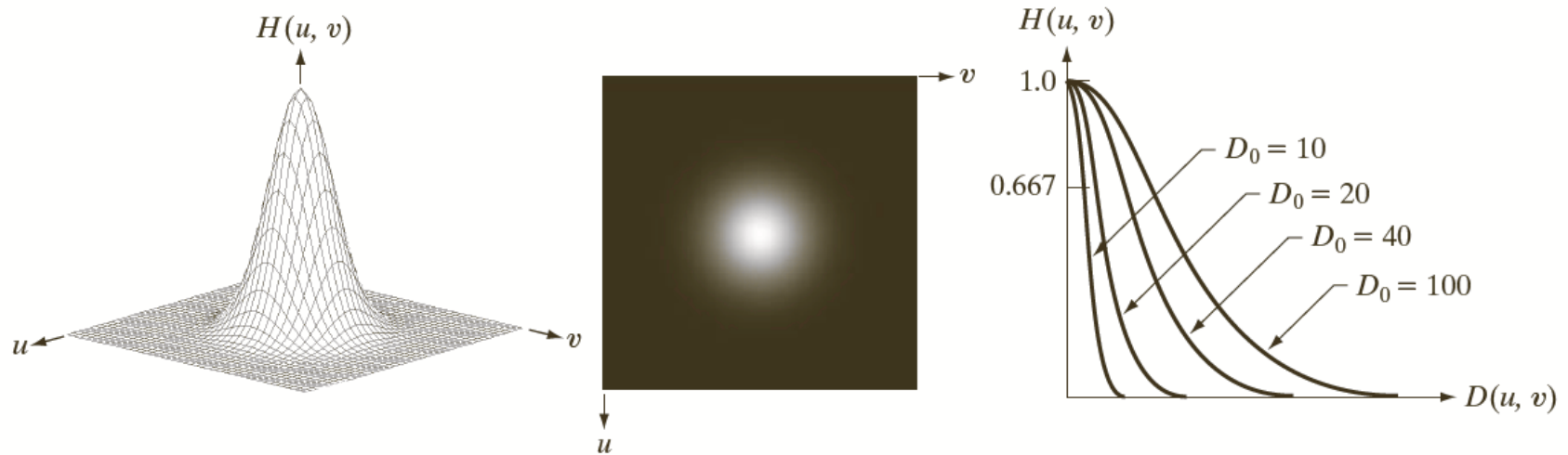


Beispiel: GLPF, $D_0 = 20$



Transferfunktion des GLPF

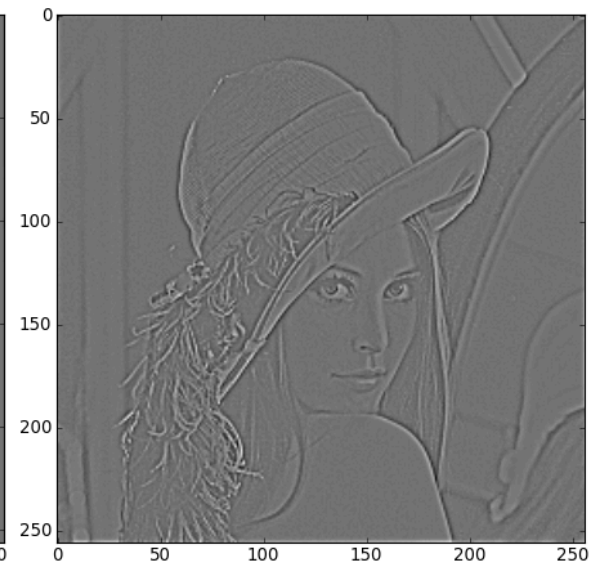
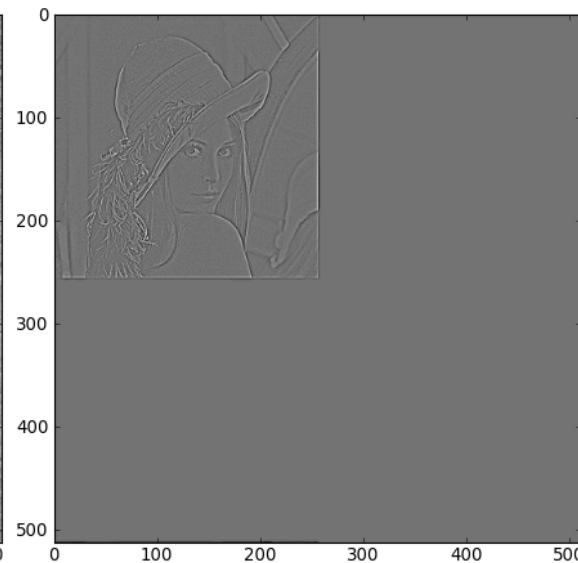
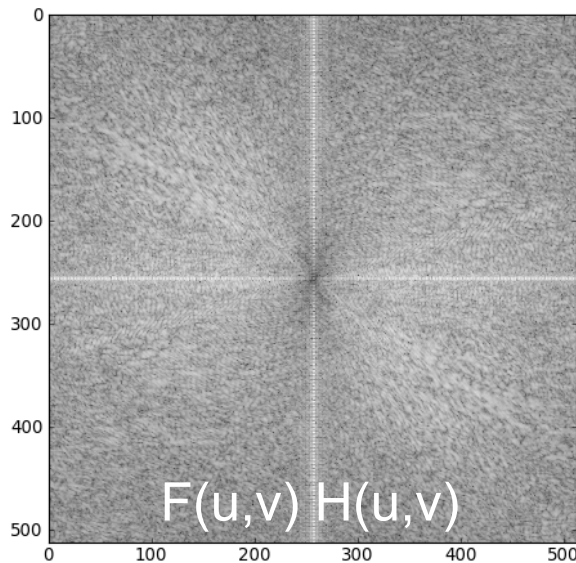
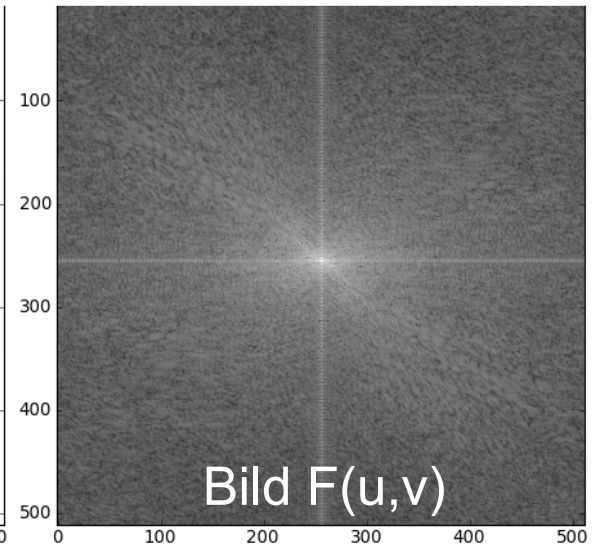
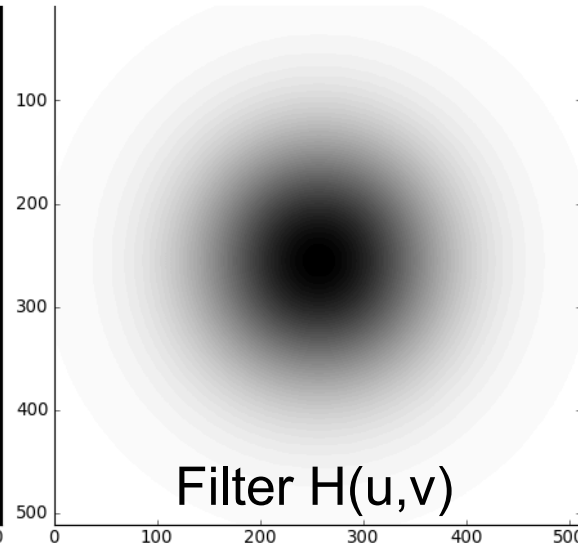
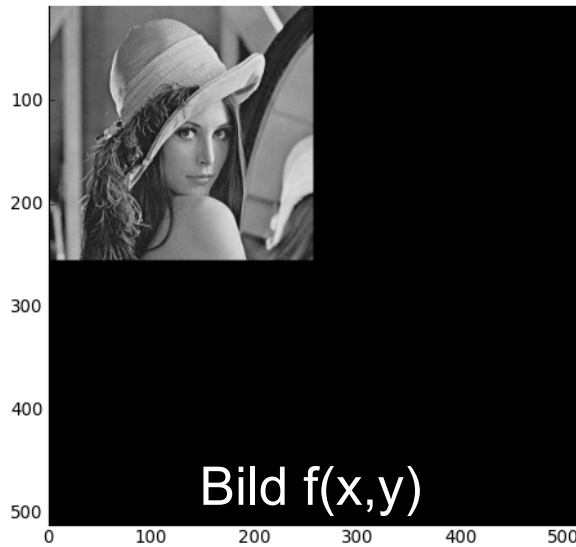
- GLPF begrenzt weniger scharf, als das BLPF
- GLPF erzeugt keine Ringing-Artefakte



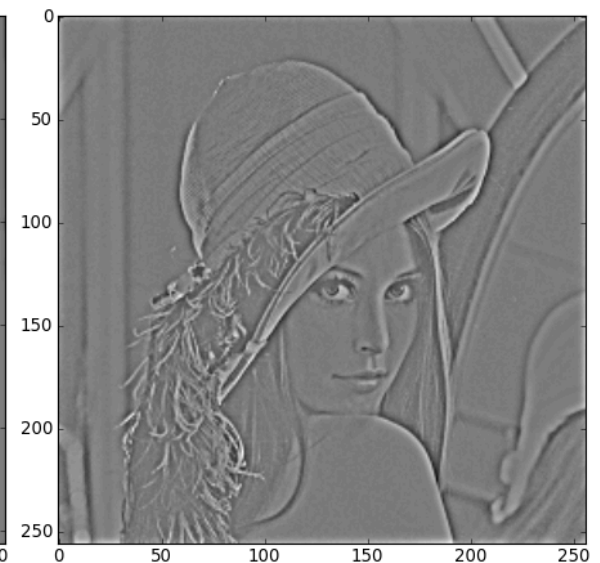
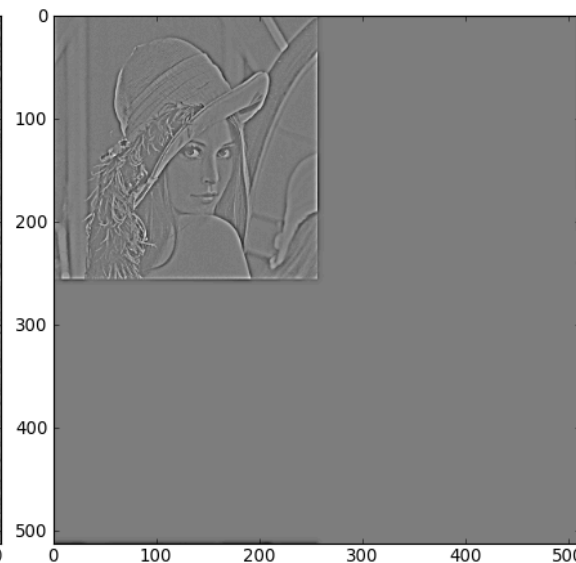
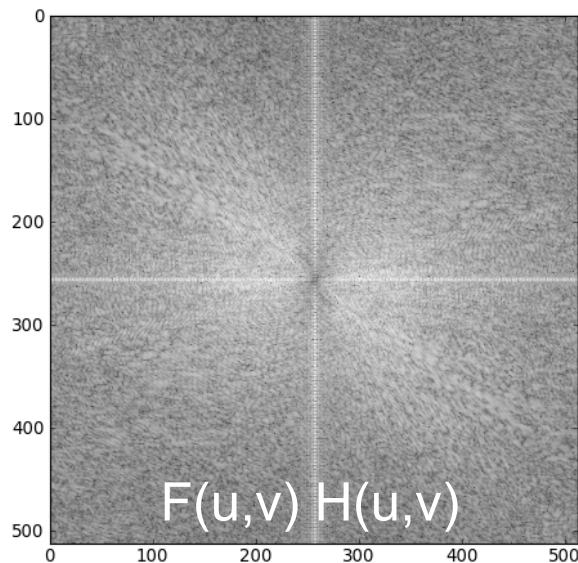
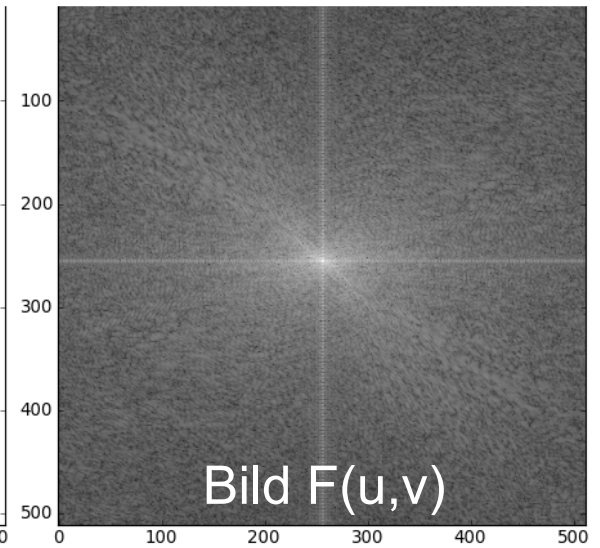
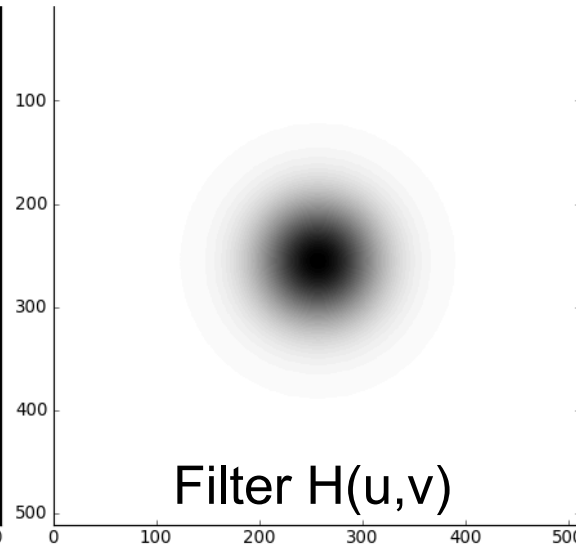
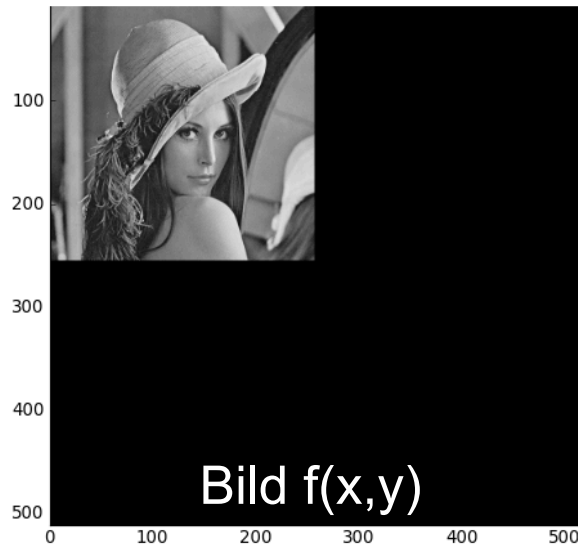
a b c

FIGURE 4.47 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0 .

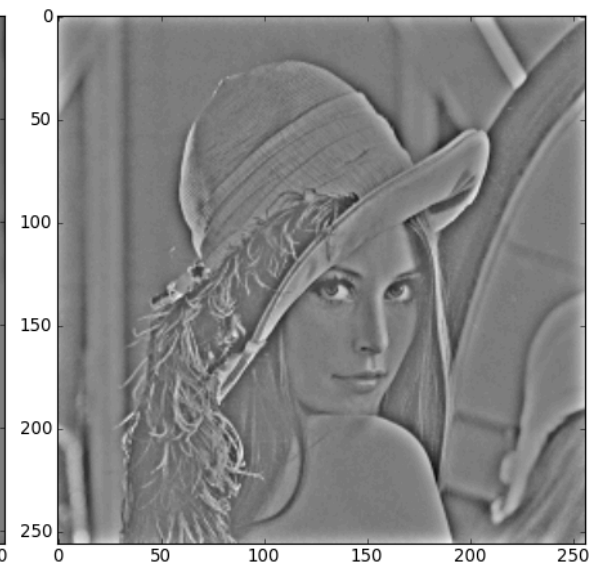
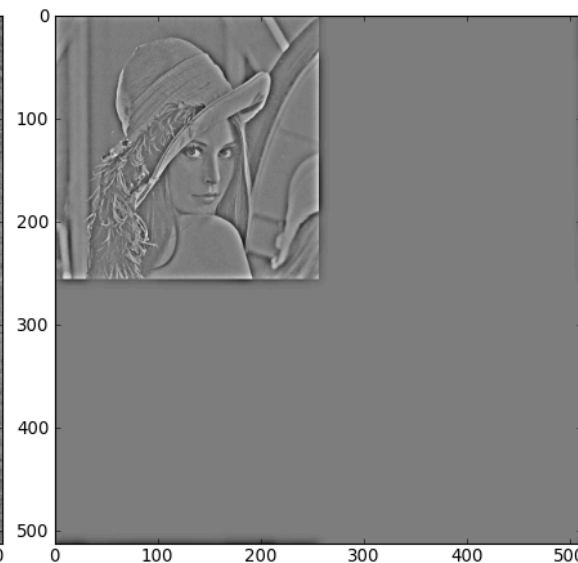
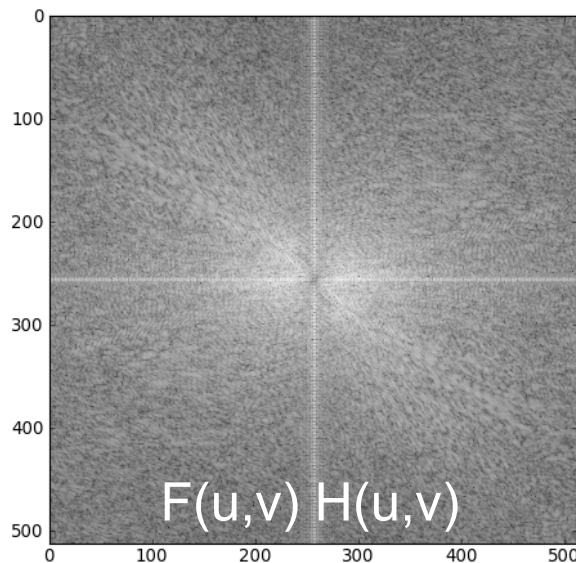
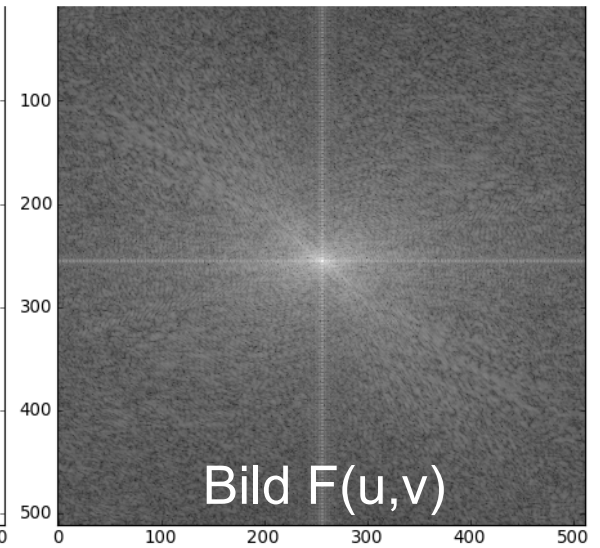
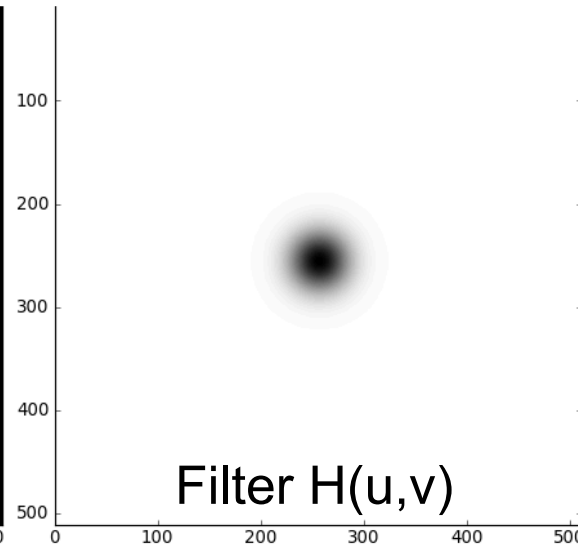
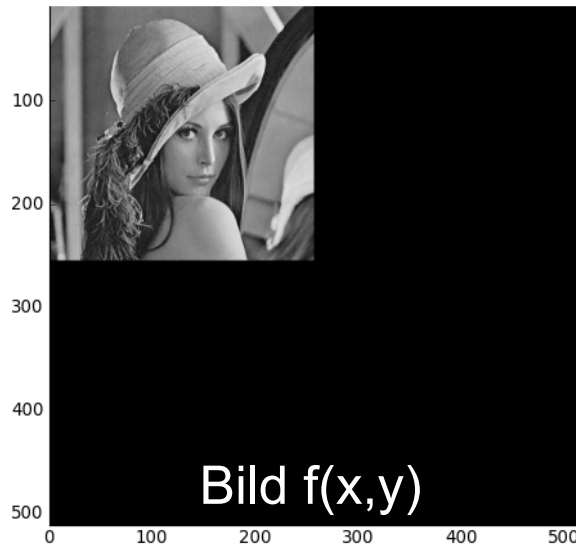
Beispiel: GHPF, $D_0 = 80$



Beispiel: GHPF, $D_0 = 40$



Beispiel: GHPF, $D_0 = 20$



Vermeiden der Auslöschung niedriger Frequenzen beim GHPF

- Hochpass-Filter (GHPF):

$$H(u, v) = 1 - \exp\left(-\frac{D^2(u, v)}{2D_0^2}\right)$$

$0 \leq H(u, v) \leq 1$

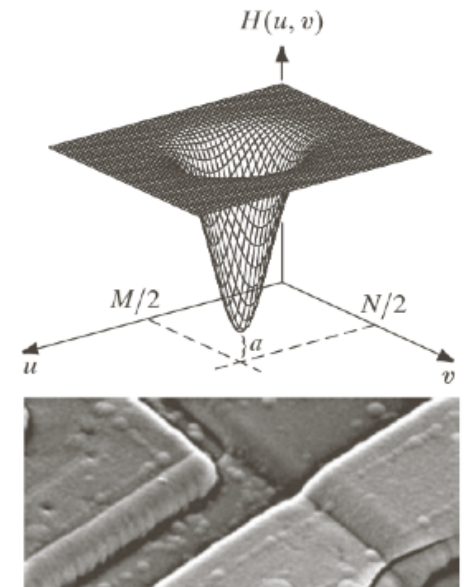
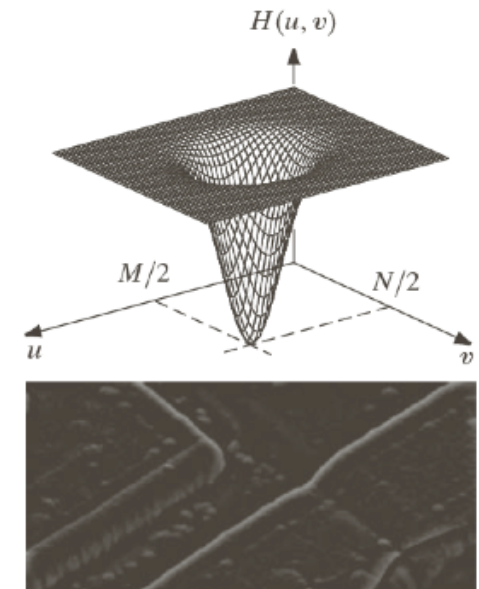
- modifiziertes GHPF:

$$H_a(u, v) = a + (1 - a) \cdot H(u, v)$$

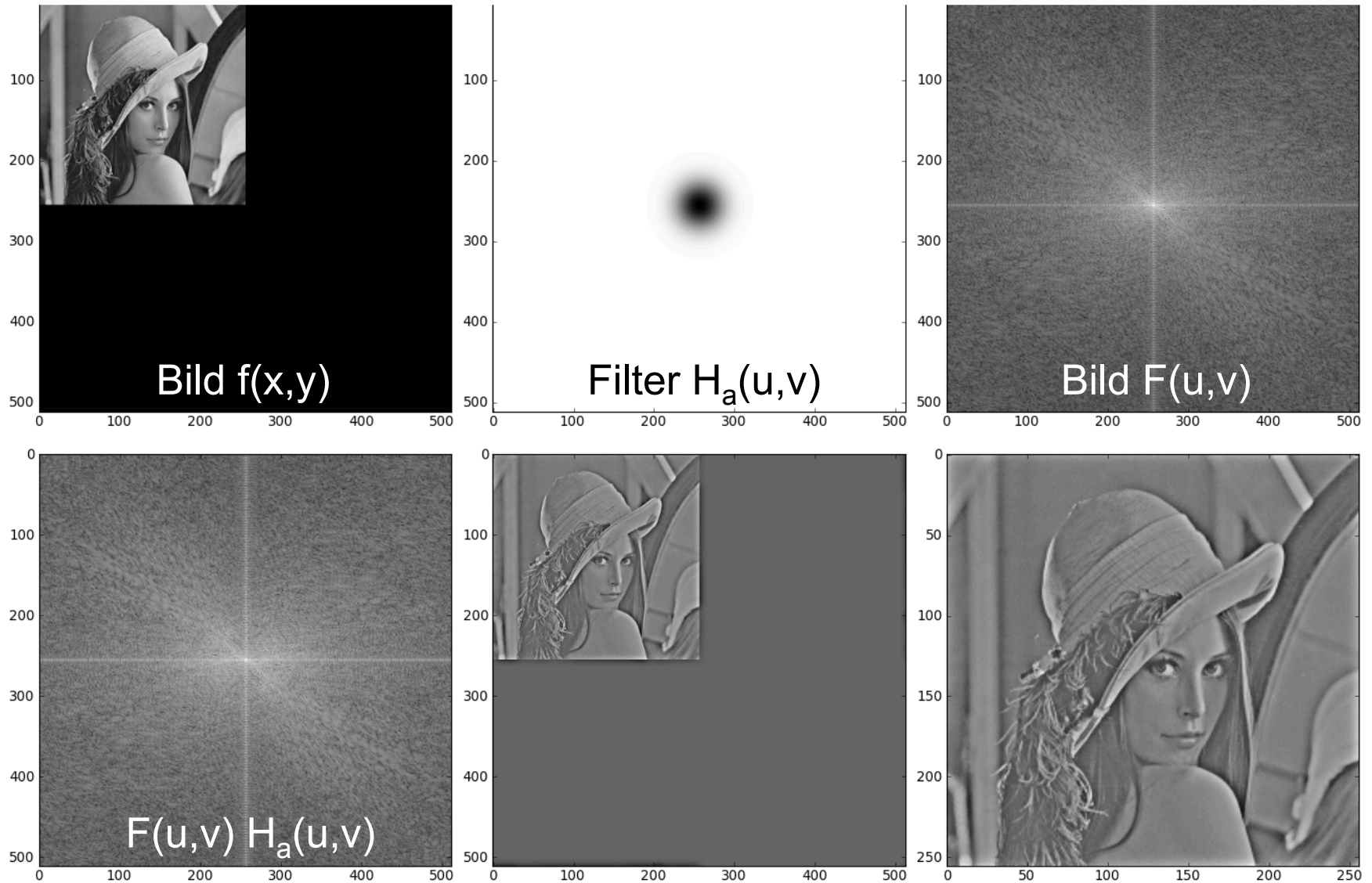
$a \leq H_a(u, v) \leq 1$

D_0 : entspricht σ

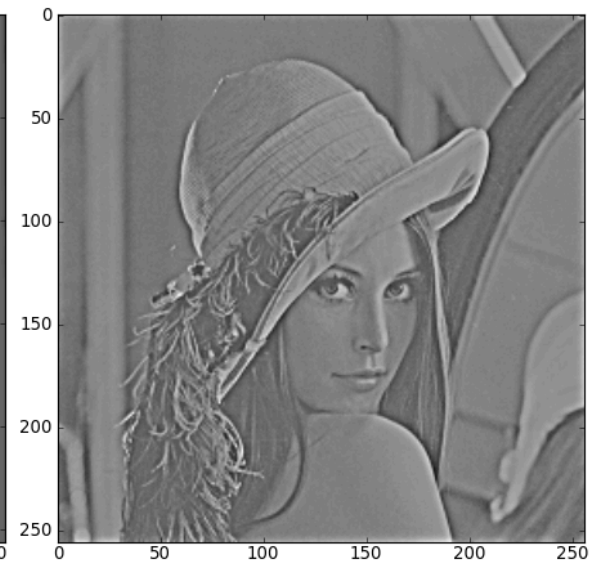
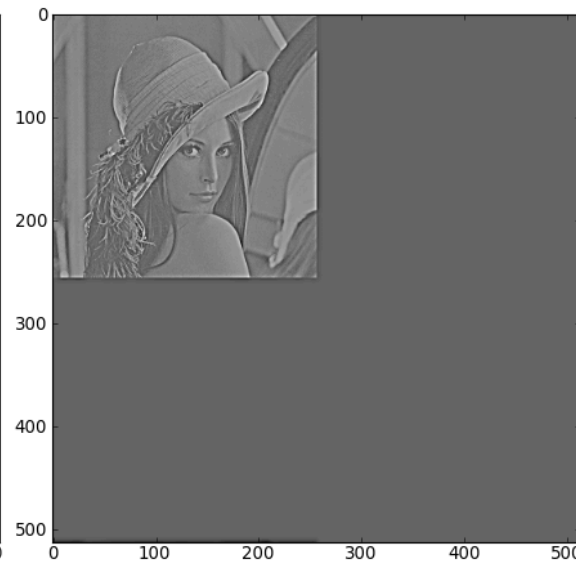
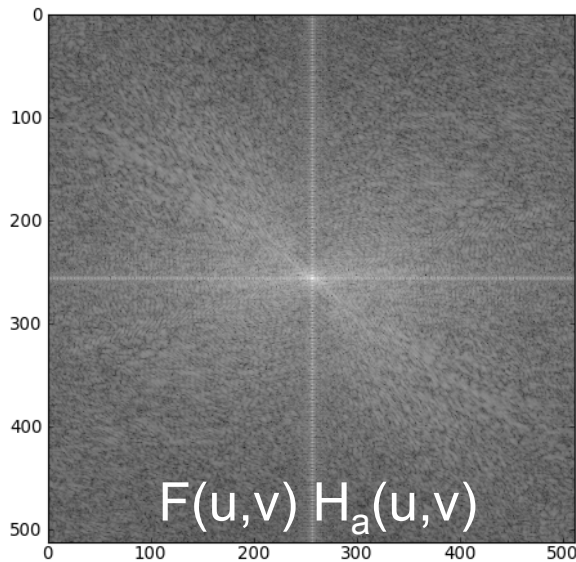
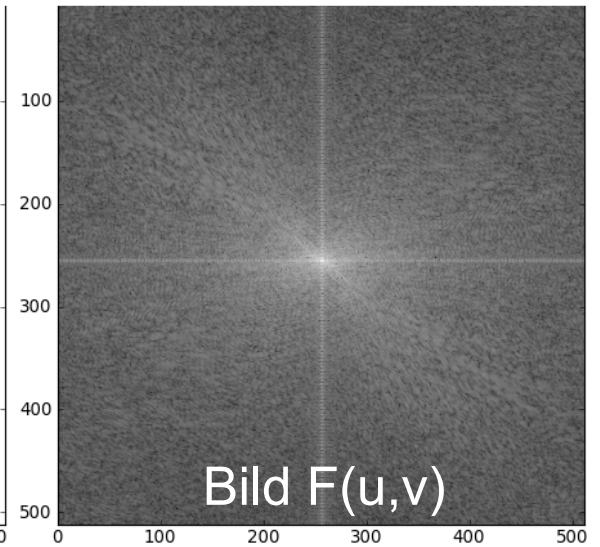
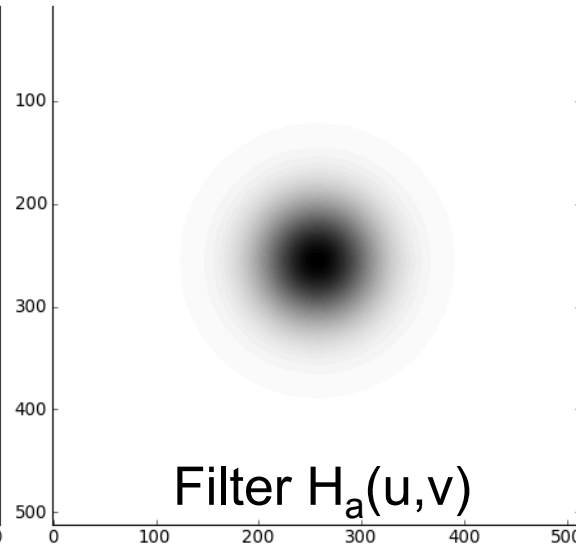
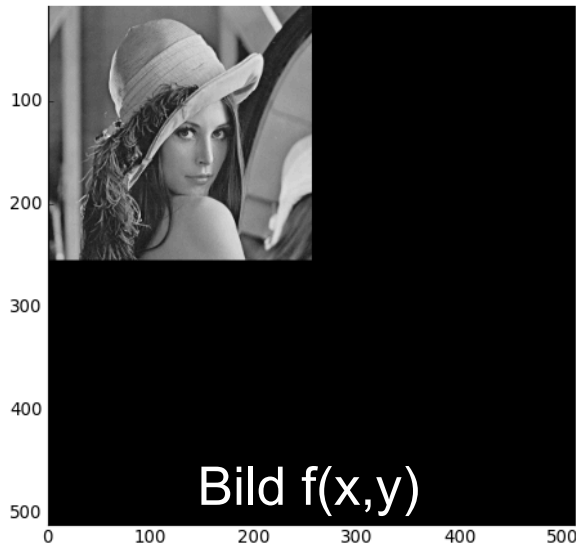
$D(u, v)$: Frequenz, Abstand vom Ursprung



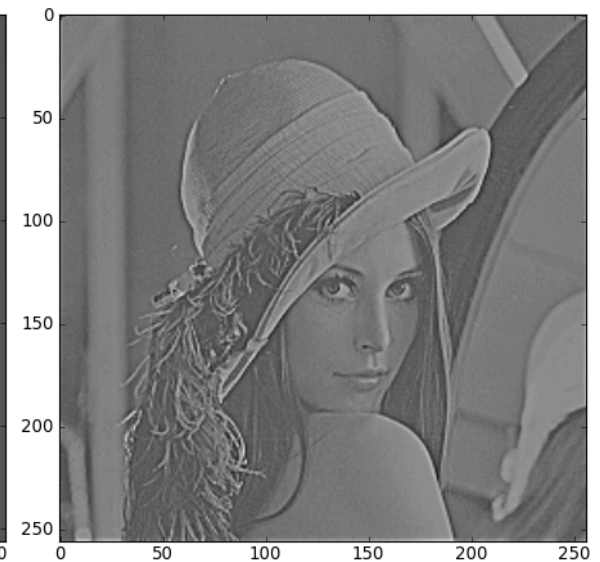
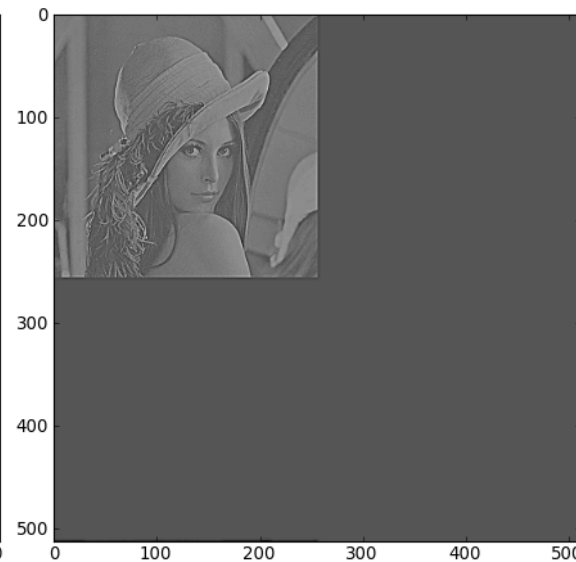
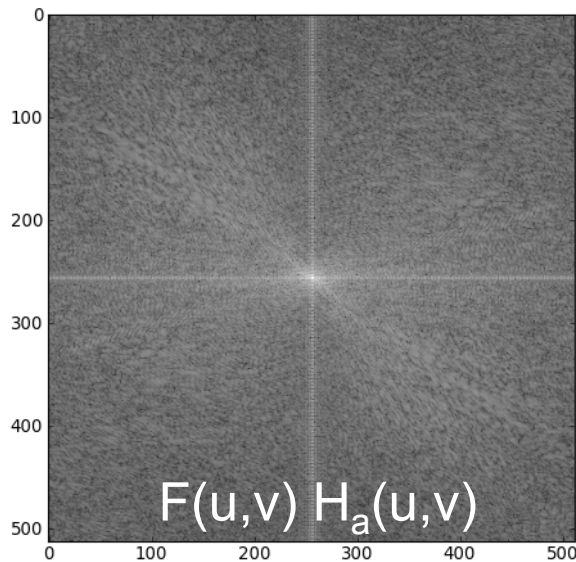
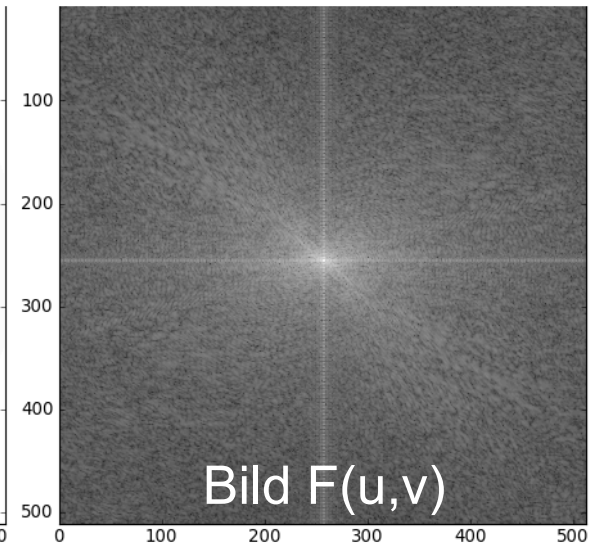
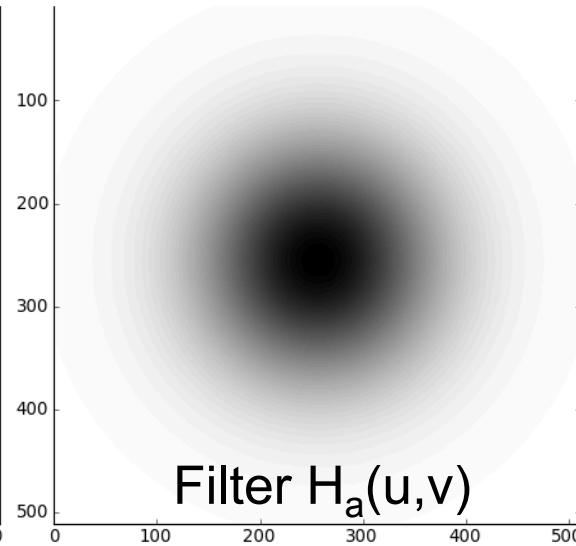
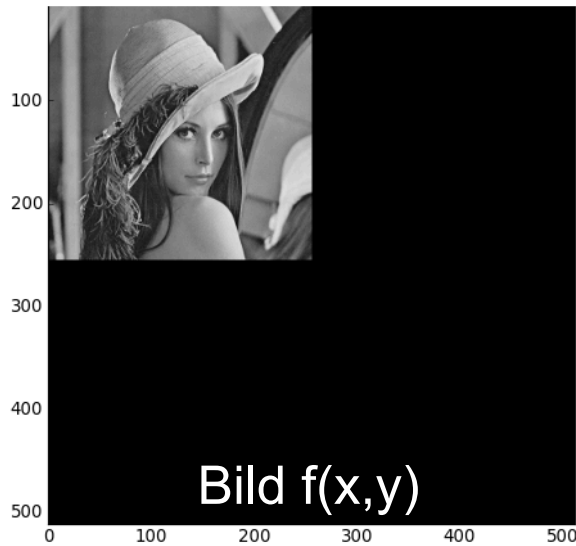
Beispiel: GHPF, $D_0 = 20$, $H_a(u,v)$, $a = 0.2$



Beispiel: GHPF, $D_0 = 40$, $H_a(u,v)$, $a = 0.2$



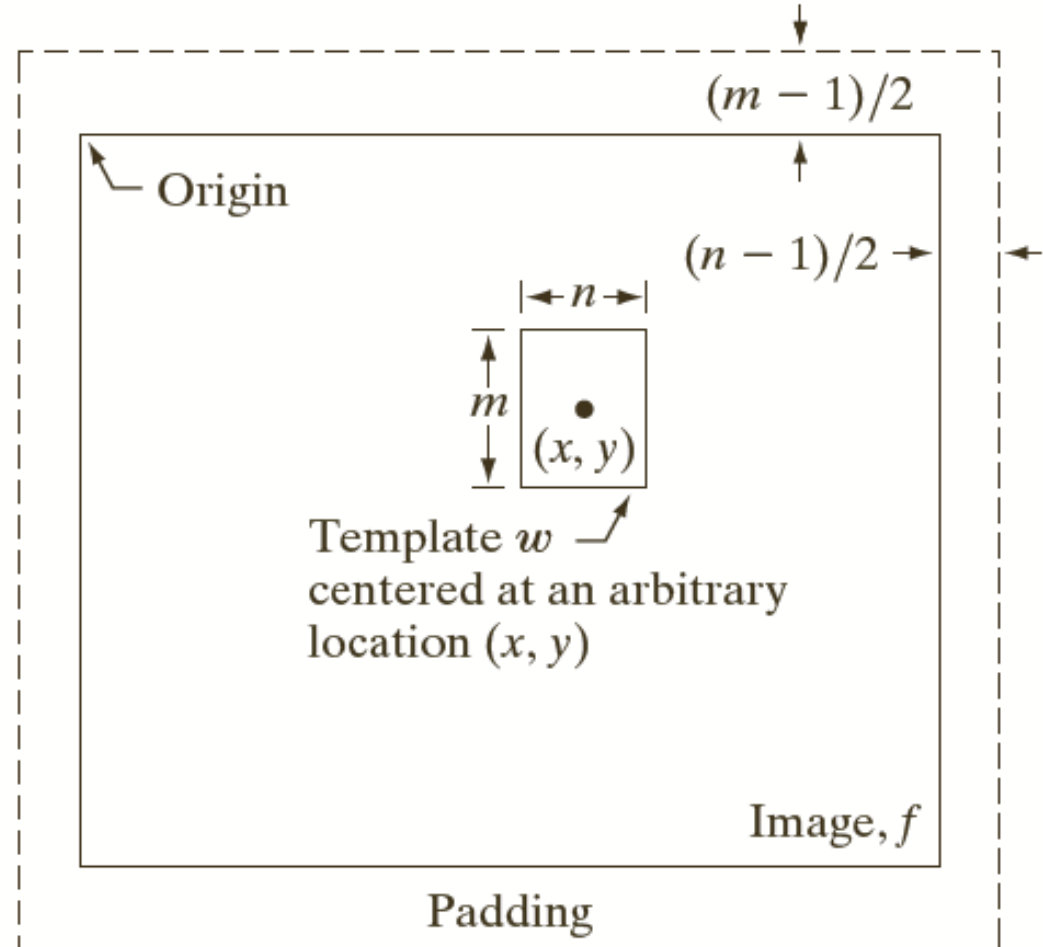
Beispiel: GHPF, $D_0 = 80$, $H_a(u,v)$, $a = 0.2$



KORRELATION IM FREQUENZRAUM

Korrelation im Ortsraum

- Ähnlichkeiten zwischen Bild und Modell feststellen
- Modell (Template) im Bild suchen



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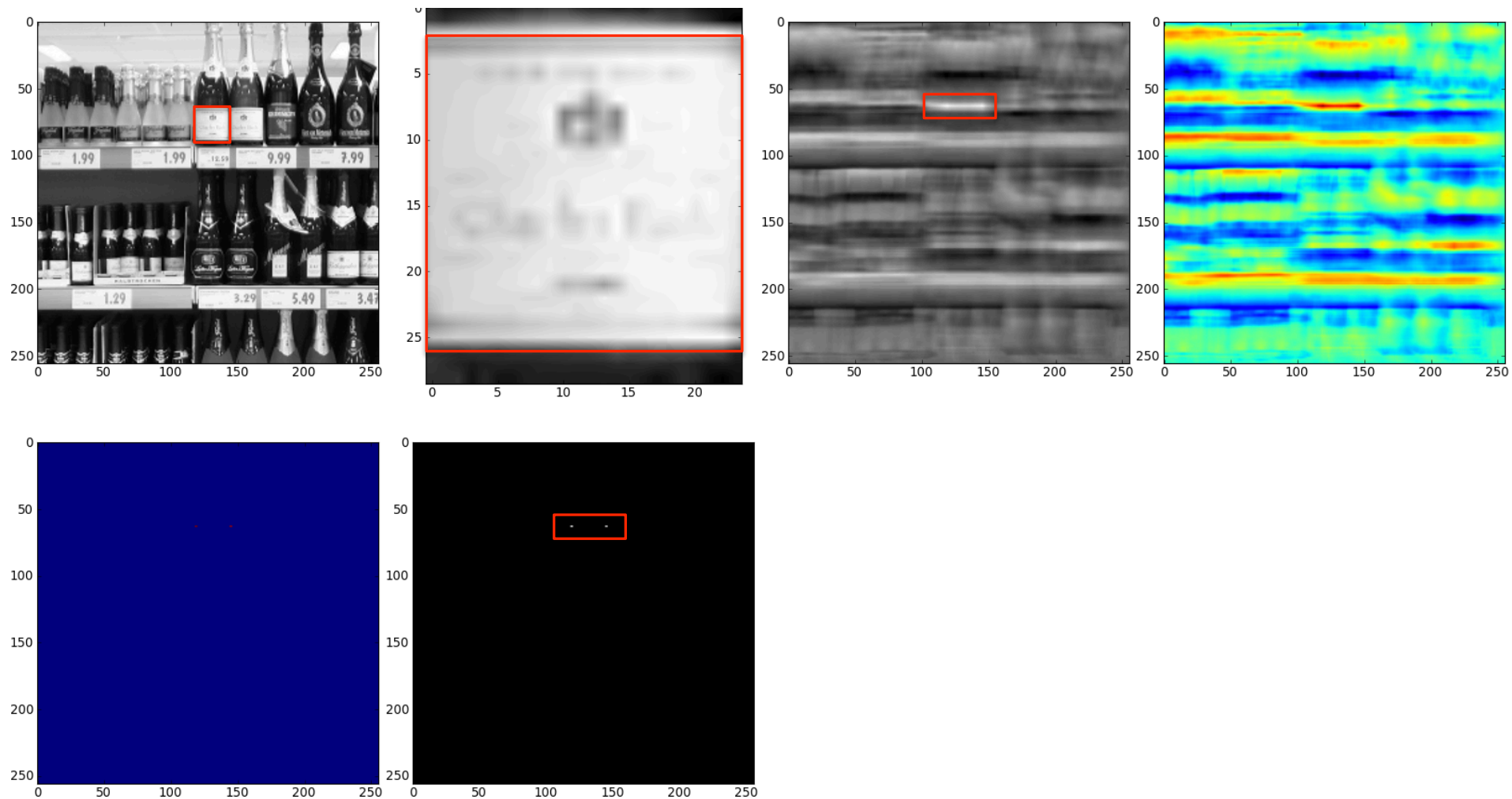
Korrelation im Ortsraum

- Ähnlichkeitsmaß (normalisierter) Korrelationskoeffizient
 - Mittelwerte subtrahiert und Varianzen normiert
- Kleineres Bild pixelweise über größeres Bild verschieben und Korrelationskoeffizient berechnen
(f = Bild, m = Modell/Template, (x,y) = Suchposition)

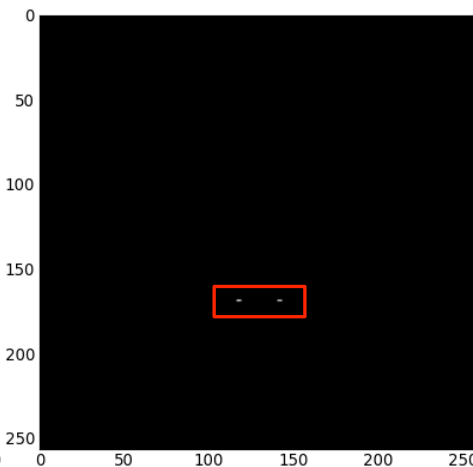
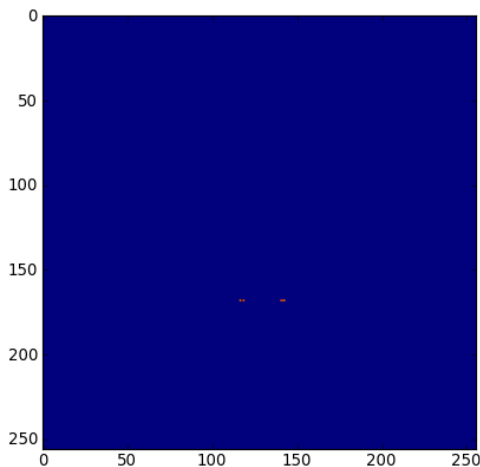
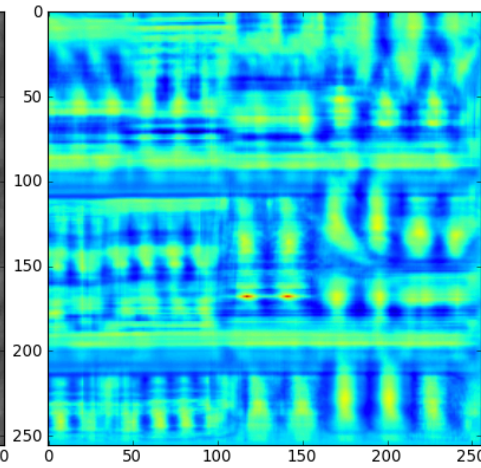
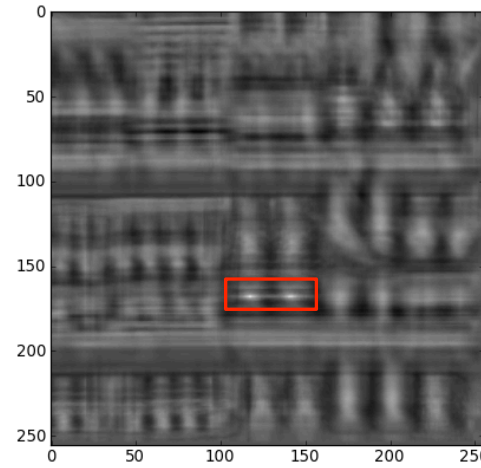
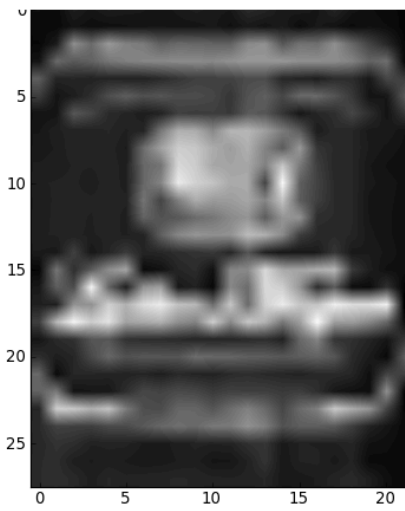
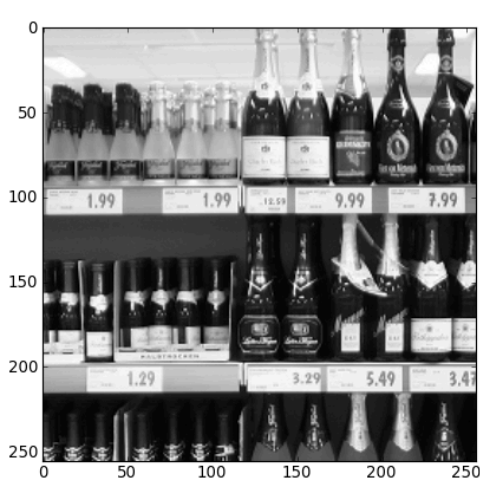
$$cc_{f,m}(x,y) = \frac{\sum_{s=-a}^a \sum_{t=-b}^b (m(s,t) - \bar{m})(f(x+s, y+t) - \bar{f}(x,y))}{\sqrt{\left[\sum_{s=-a}^a \sum_{t=-b}^b (m(s,t) - \bar{m})^2 \right] \left[\sum_{s=-a}^a \sum_{t=-b}^b (f(x+s, y+t) - \bar{f}(x,y))^2 \right]}}$$

$$-1 \leq cc_{f,m}(x,y) \leq 1$$

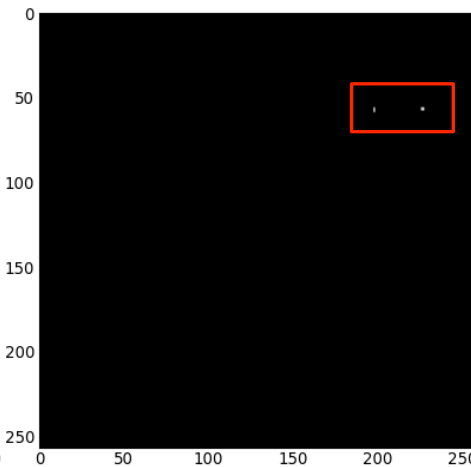
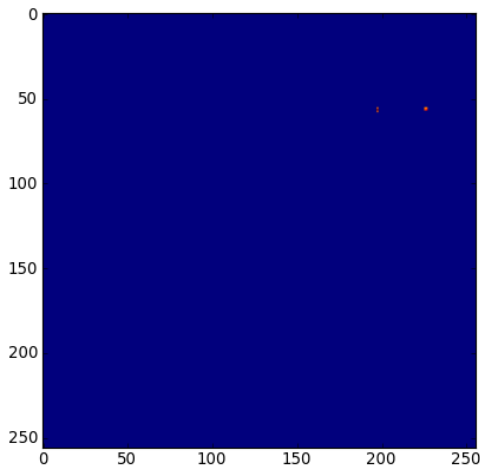
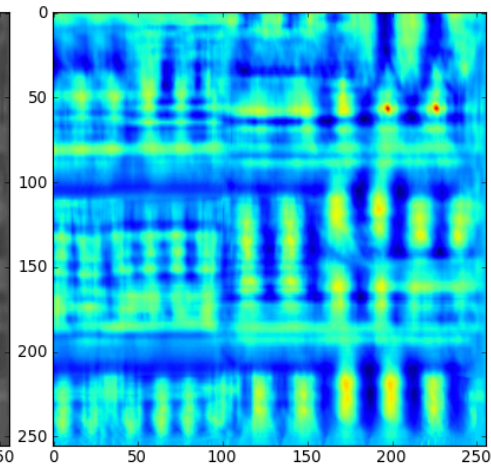
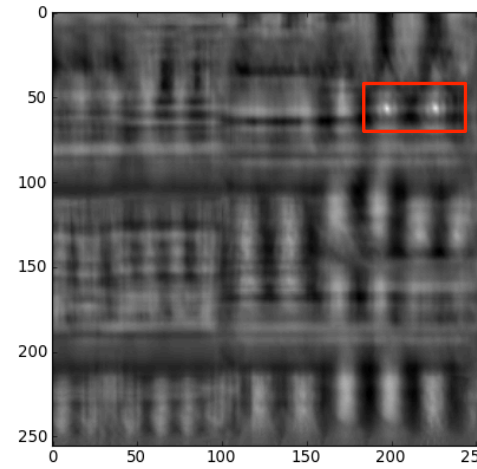
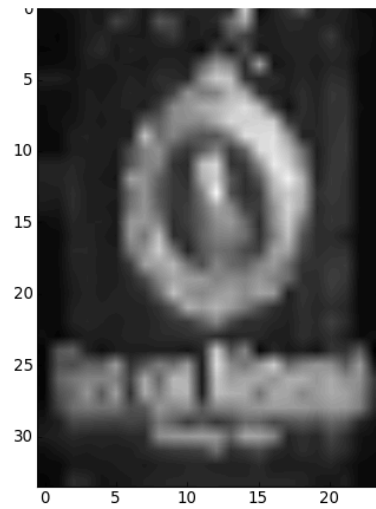
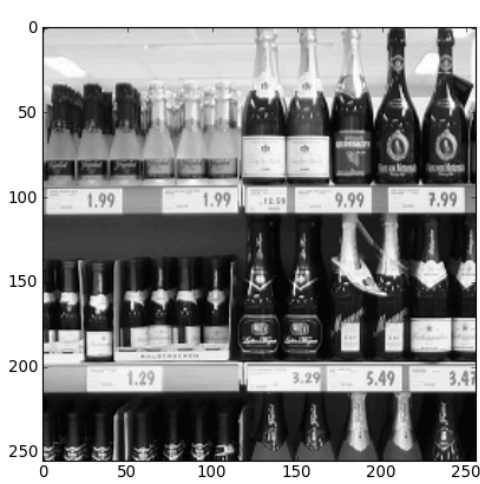
Korrelation im Ortsraum



Korrelation im Ortsraum



Korrelation im Ortsraum



Korrelation im Frequenzraum

- Subtrahiere Mittelwerte von f und m
- Ähnlichkeitsmaß: Korrelationskoeffizient $cc_{f,g}$

$$\begin{aligned}
 cc_{f,m}(x,y) &= \frac{\sum_{s=-a}^a \sum_{t=-b}^b (m(s,t) - \bar{m})(f(x+s, y+t) - \bar{f}(x,y))}{\sqrt{\left[\sum_{s=-a}^a \sum_{t=-b}^b (m(s,t) - \bar{m})^2 \right] \left[\sum_{s=-a}^a \sum_{t=-b}^b (f(x+s, y+t) - \bar{f}(x,y))^2 \right]}} \\
 &= k \sum_{s=-a}^a \sum_{t=-b}^b (m(s,t) - \bar{m})(f(x+s, y+t) - \bar{f}(x,y)) \\
 &= k \underbrace{\sum_{s=-a}^a \sum_{t=-b}^b m(s,t) f(x+s, y+t)}_{\text{Korrelationsfunktion}} \implies FT([f \circ g](x,y)) = F(u,v) \cdot G^*(u,v) \\
 &\hspace{15em} \text{Korrelation} \hspace{15em} \text{G konjugiert}
 \end{aligned}$$

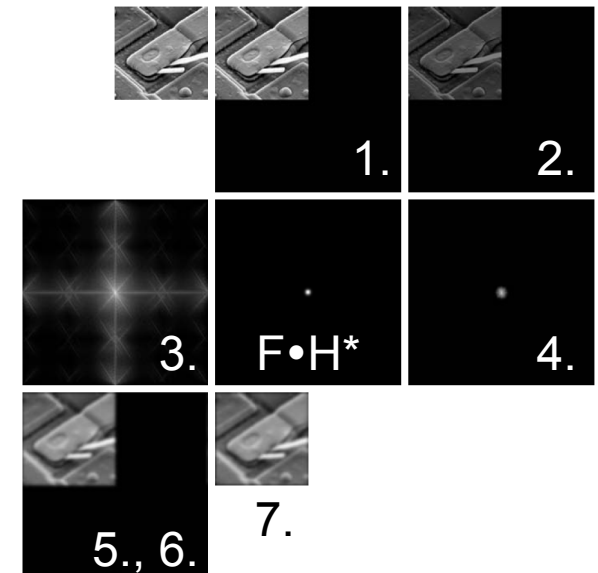
Korrelation im Frequenzraum

- Statt Korrelation im Ortsraum nun Multiplikation mit konjugiert komplexem Modell im Frequenzraum

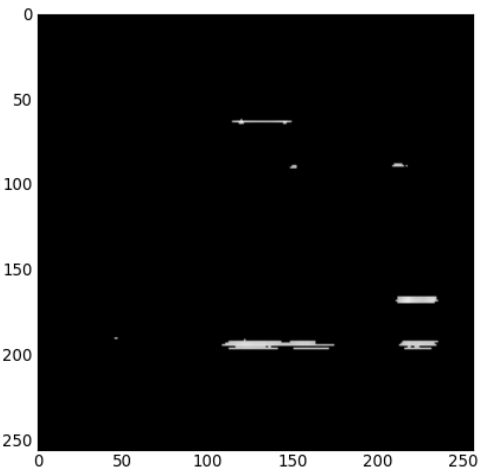
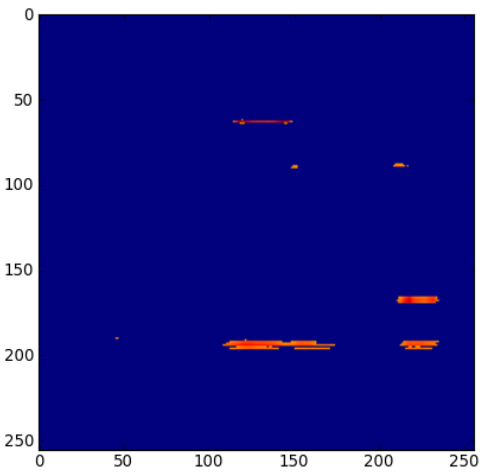
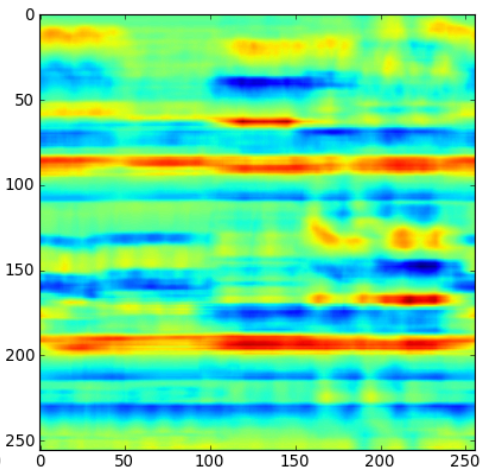
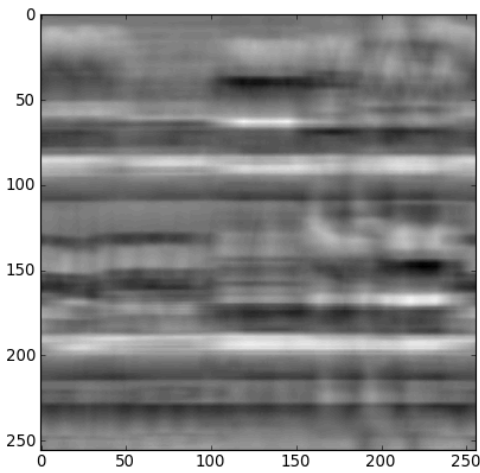
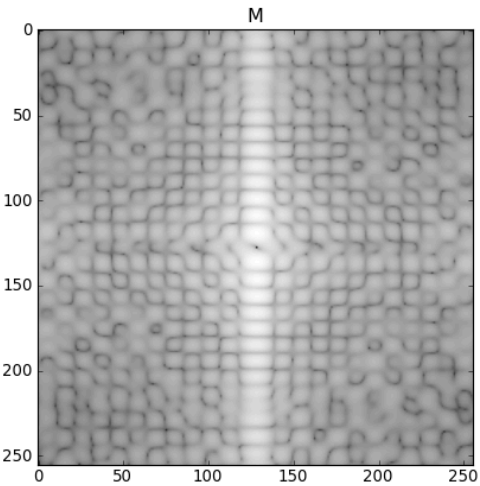
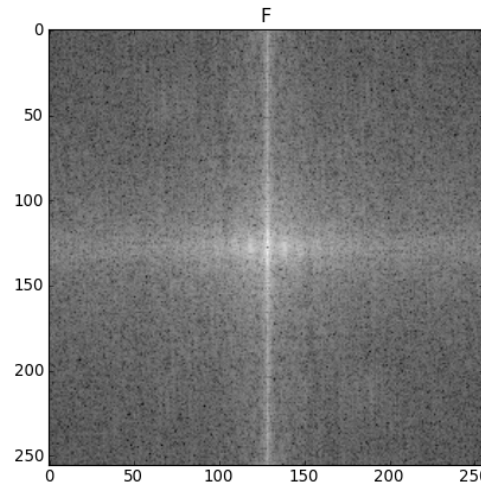
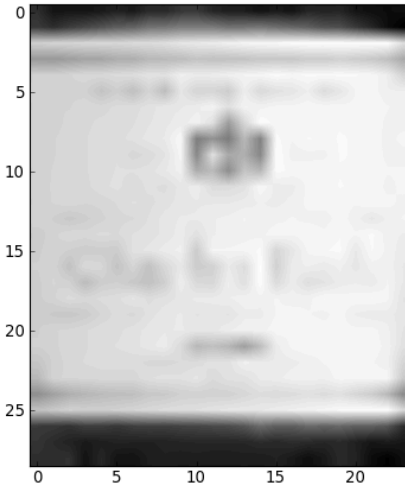
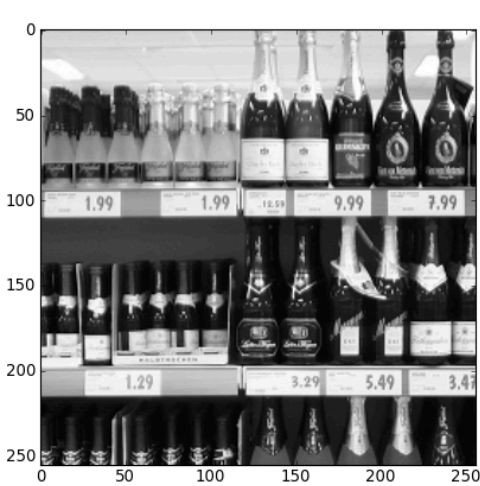
$$f(x, y) \circ h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) \cdot h(x+m, y+n) \Leftrightarrow F(u, v) \cdot H^*(u, v)$$

- Vorgehen

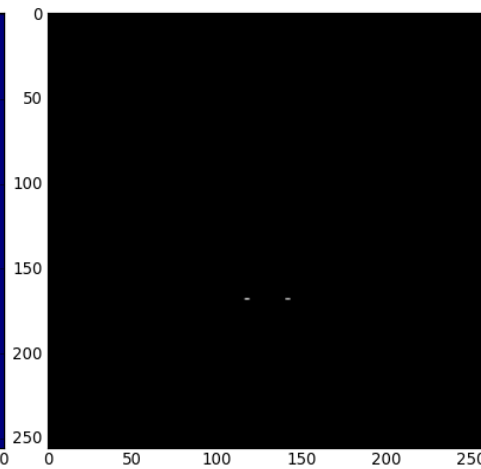
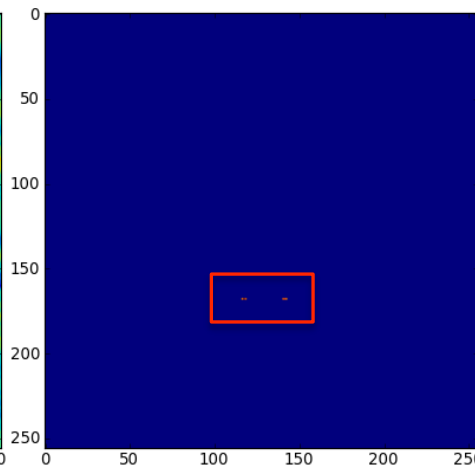
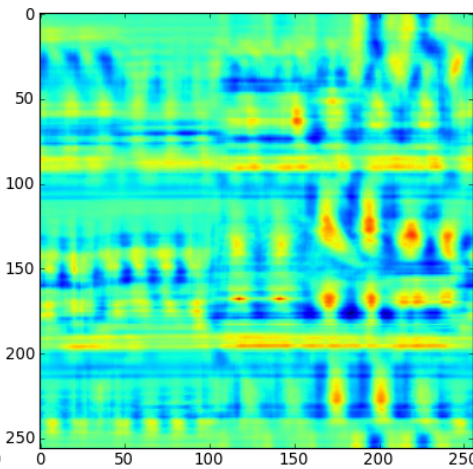
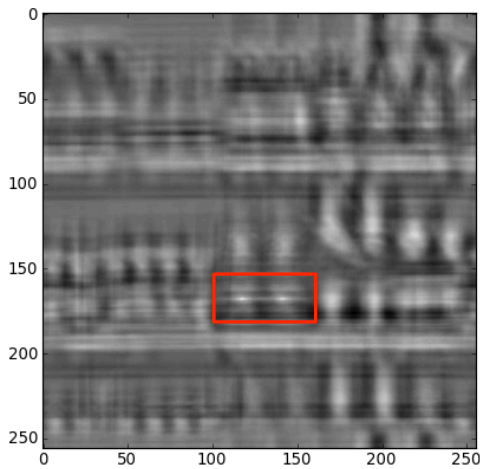
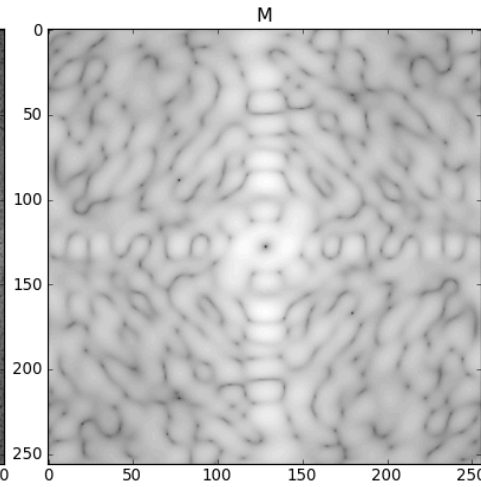
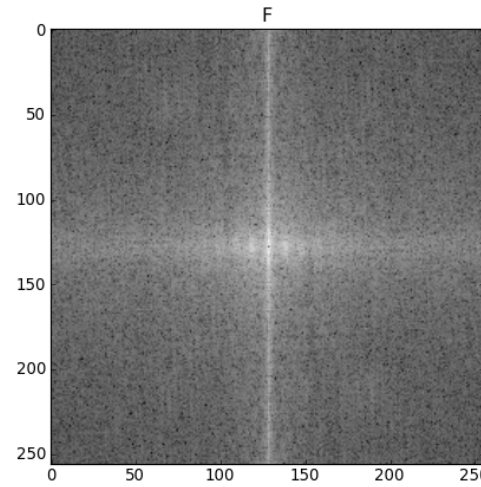
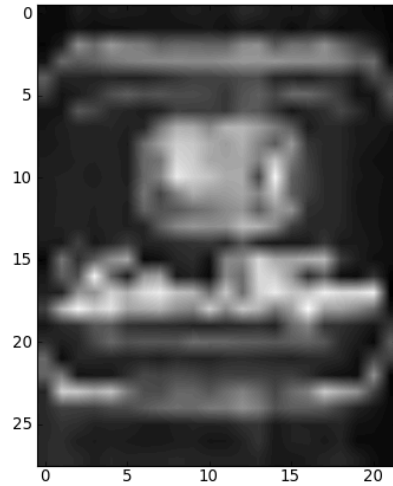
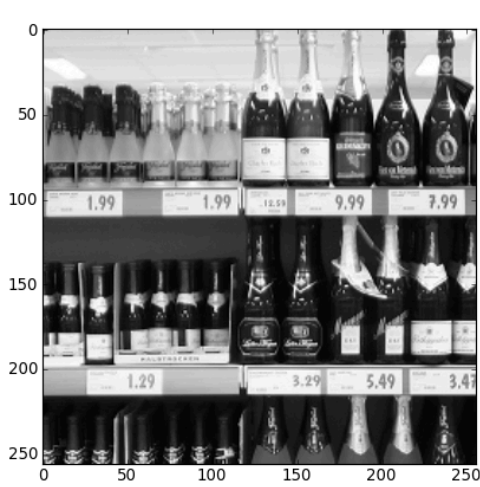
1. Mittelwerte abziehen
2. Padding im Ortsraum auf $2M, 2N$
3. Multiplikation mit $(-1)^{x+y}$
4. DFT
5. Multiplikation mit H^* im Frequenzraum
6. inverse DFT, Realteil
7. Multiplikation mit $(-1)^{x+y}$
8. Padding entfernen



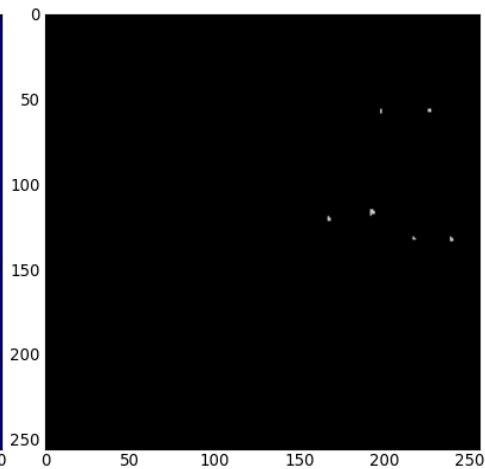
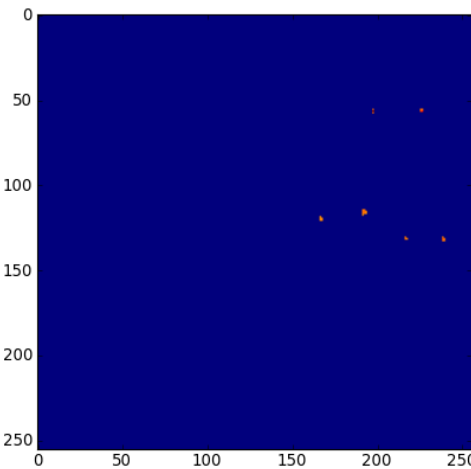
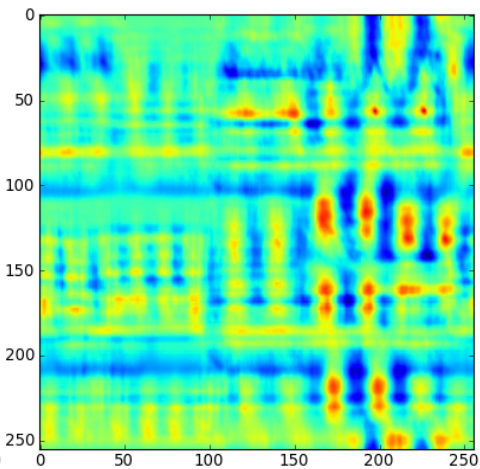
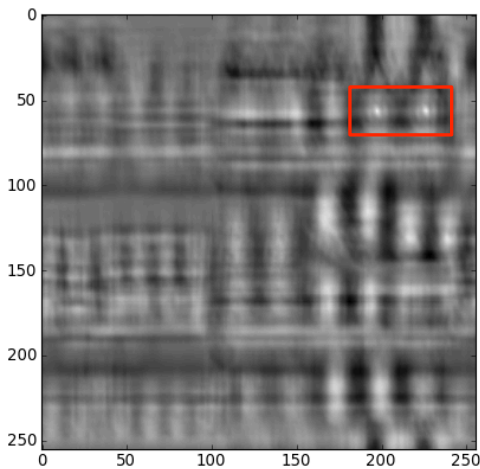
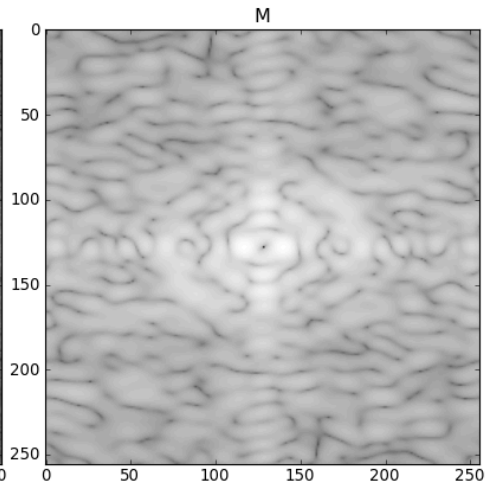
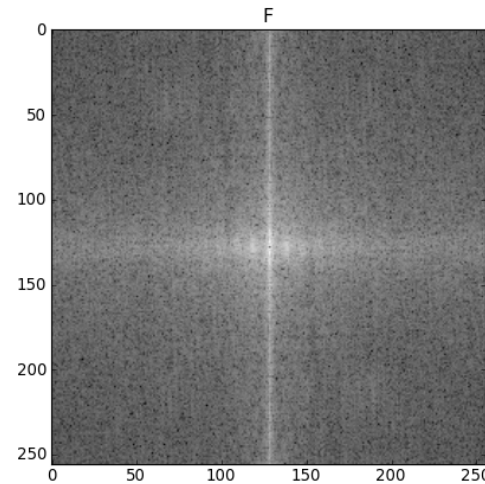
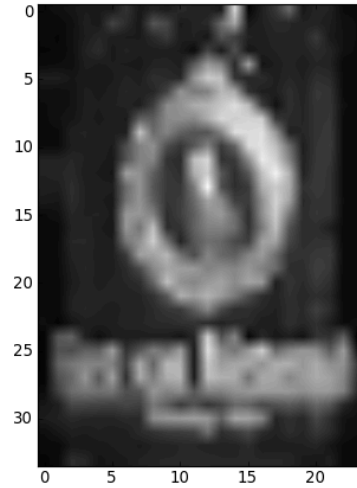
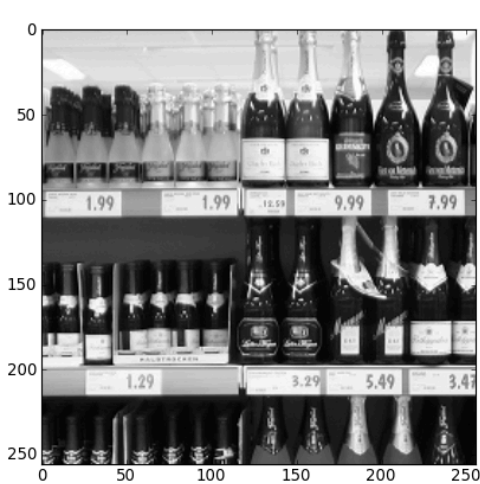
Korrelation im Frequenzraum



Korrelation im Frequenzraum



Korrelation im Frequenzraum



Grenzen des Korrelations-Matching

- Abhängigkeit von
 - Skalierung
 - Rotation
 - Perspektive
- später in der Vorlesung leistungsfähigere Verfahren

Bandreject/Bandpass-Filter

Notch-Filter

Bandreject-Filter

- Nützlich, wenn Störungen in einem bestimmten Frequenzband konzentriert sind
- Selektive Bandreject-Filter können solche Störungen reduzieren



FIGURE 5.15 From left to right, perspective plots of ideal, Butterworth (of order 1), and Gaussian bandreject filters.

Bandreject/Bandpass-Filter

- Bandreject-Filter

- Ideal
$$H(u, v) = \begin{cases} 0 & , \text{ falls } D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \\ 1 & , \text{ sonst} \end{cases}$$

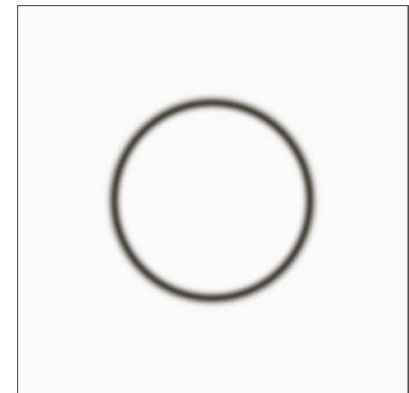
- Butterworth
$$H(u, v) = \frac{1}{1 + \left(\frac{DW}{D^2 - D_0^2} \right)^{2n}}$$

- Gauß
$$H(u, v) = 1 - \exp\left(-\left(\frac{D^2 - D_0^2}{DW}\right)^2\right)$$

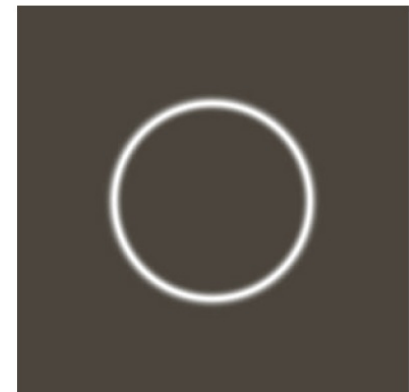
- Bandpass-Filter

$$H_{BP}(u, v) = 1 - H_{BR}(u, v)$$

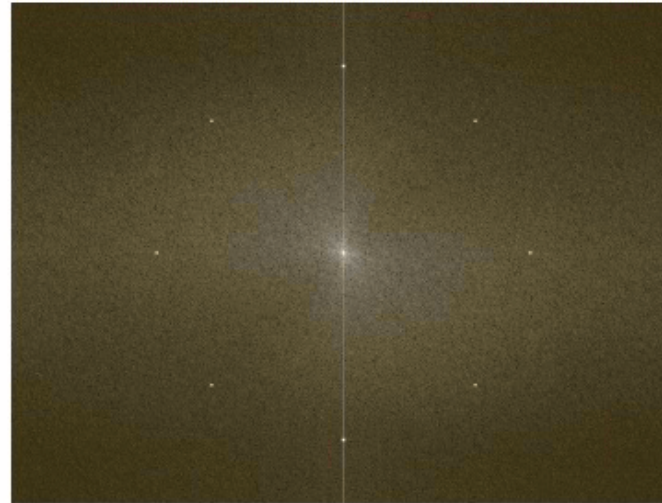
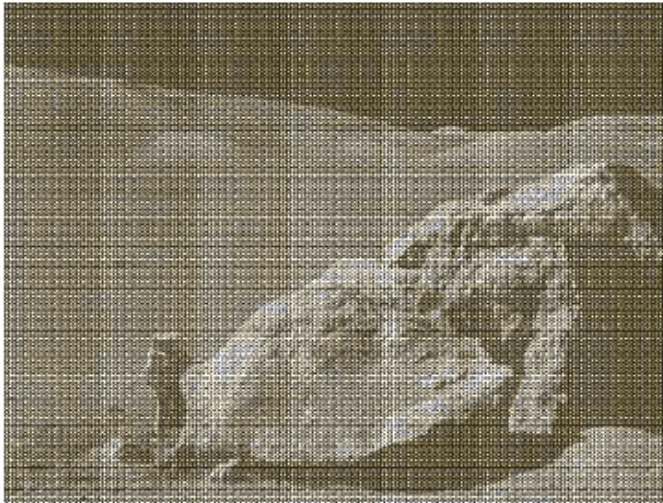
Gauß-
Bandreject-
Filter



Gauß-
Bandpass-
Filter



Beispiel: Artefakte entfernen



a b
c d

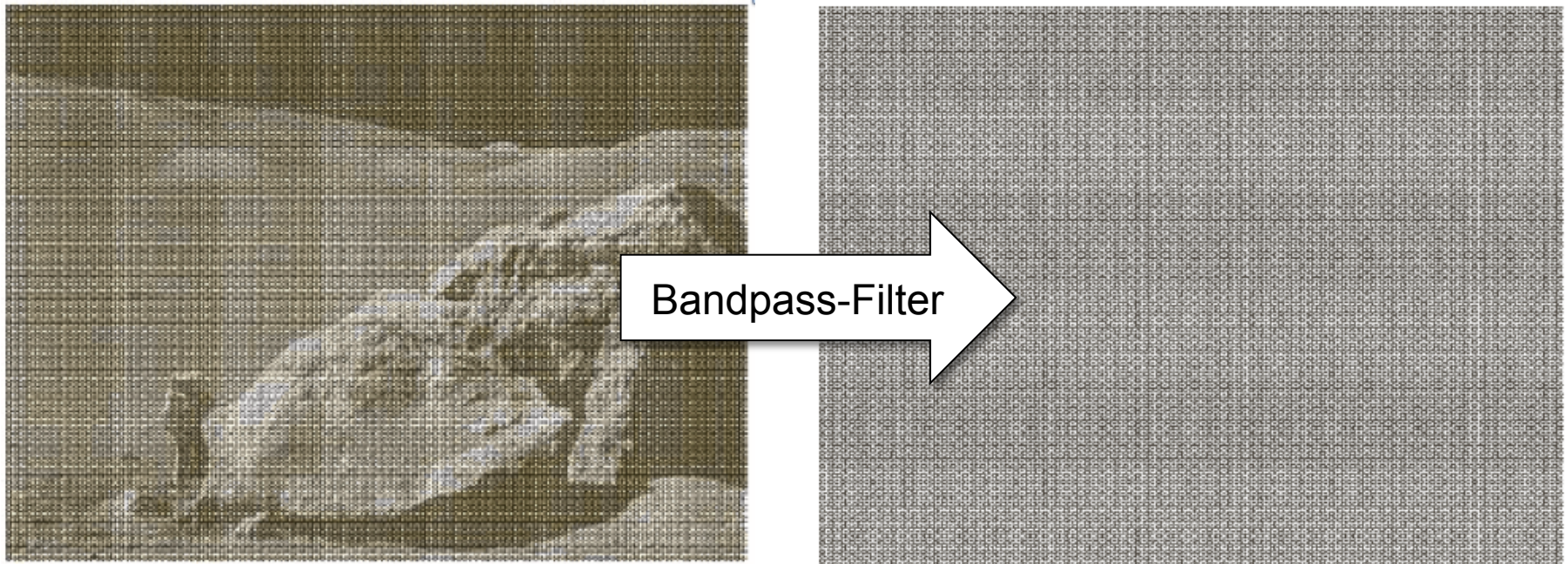
FIGURE 2.40

(a) Image corrupted by sinusoidal interference. (b) Magnitude of the Fourier transform showing the bursts of energy responsible for the interference. (c) Mask used to eliminate the energy bursts. (d) Result of computing the inverse of the modified Fourier transform. (Original image courtesy of NASA.)



Beispiel: Artefakte entfernen

- Bandpassfilter isoliert Artefakt



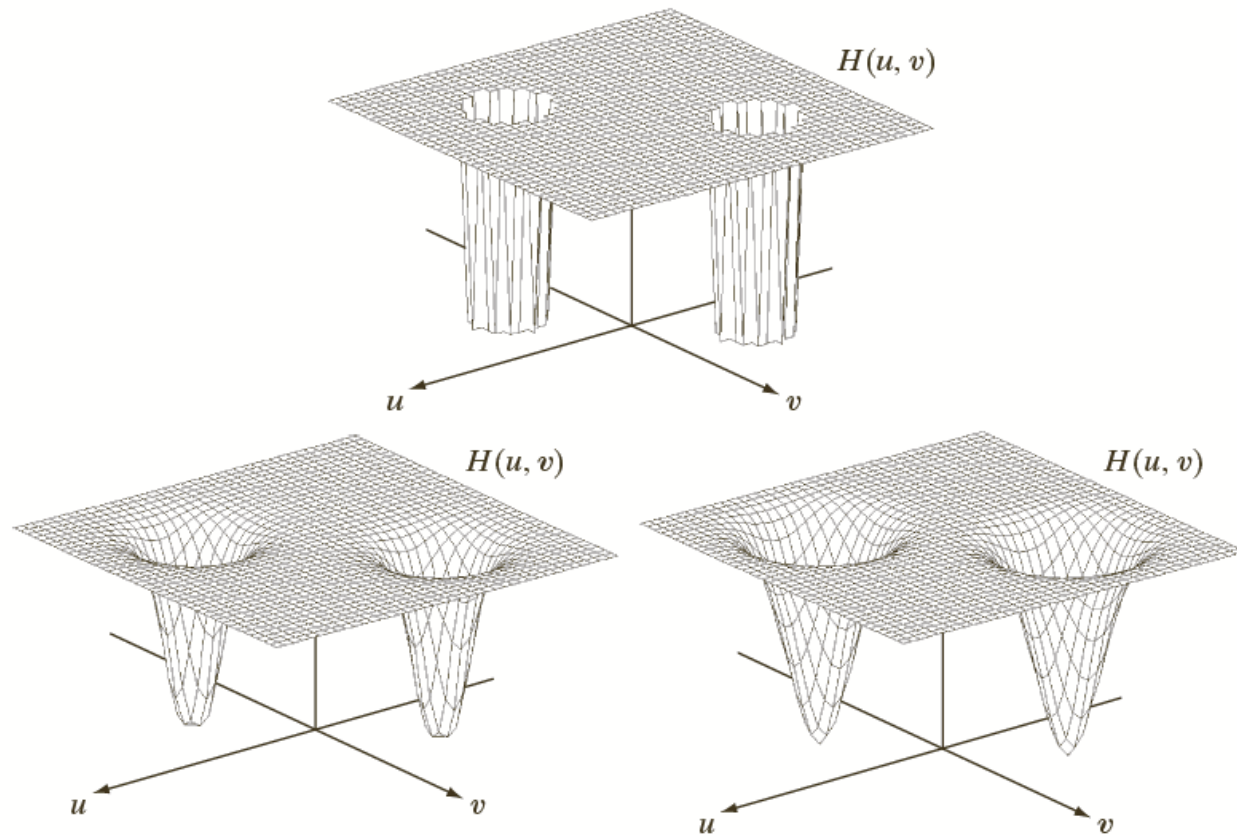
Notch-Filter

- Filtert in einer bestimmten Umgebung
- Symmetrisch zum Ursprung

a
b c

FIGURE 5.18

Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.



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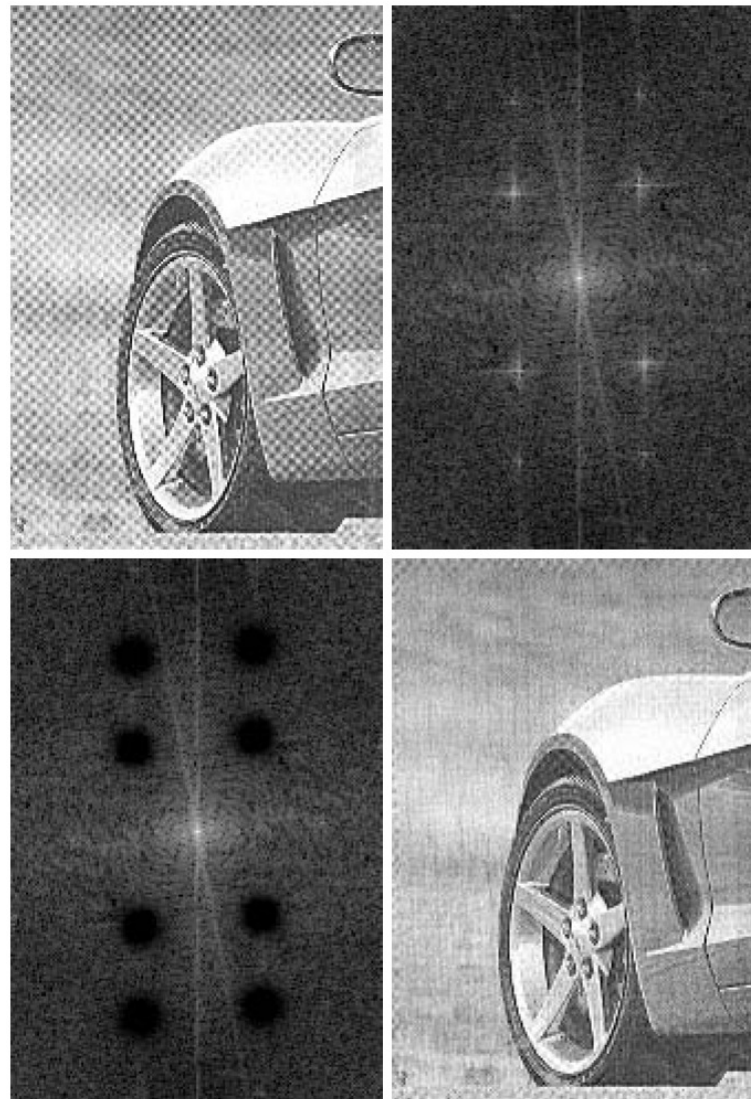
Notch-Filter

- Filtert in einer bestimmten Umgebung (u_0, v_0)
- Müssen symmetrisch zum Ursprung sein
 - wenn Zentrum (u_0, v_0) , dann auch $(-u_0, -v_0)$
 - sonst verändert sich die Phase
- Gebildet aus Produkt von Hochpass-Filtern

$$H_{NR}(u, v) = \prod_{k=1}^Q H_k(u, v) \cdot H_{-k}(u, v)$$

- wobei $H_k(u, v)$ und $H_{-k}(u, v)$ ihre Zentren bei (u_k, v_k) bzw. $(-u_k, -v_k)$ haben; Distanzen relativ zu diesen Zentren
 - Notch-Pass-Filter
- $$H_{NP}(u, v) = 1 - H_{NR}(u, v)$$

Beispiel: Reduzierung Moiré-Muster



a	b
c	d

FIGURE 4.64

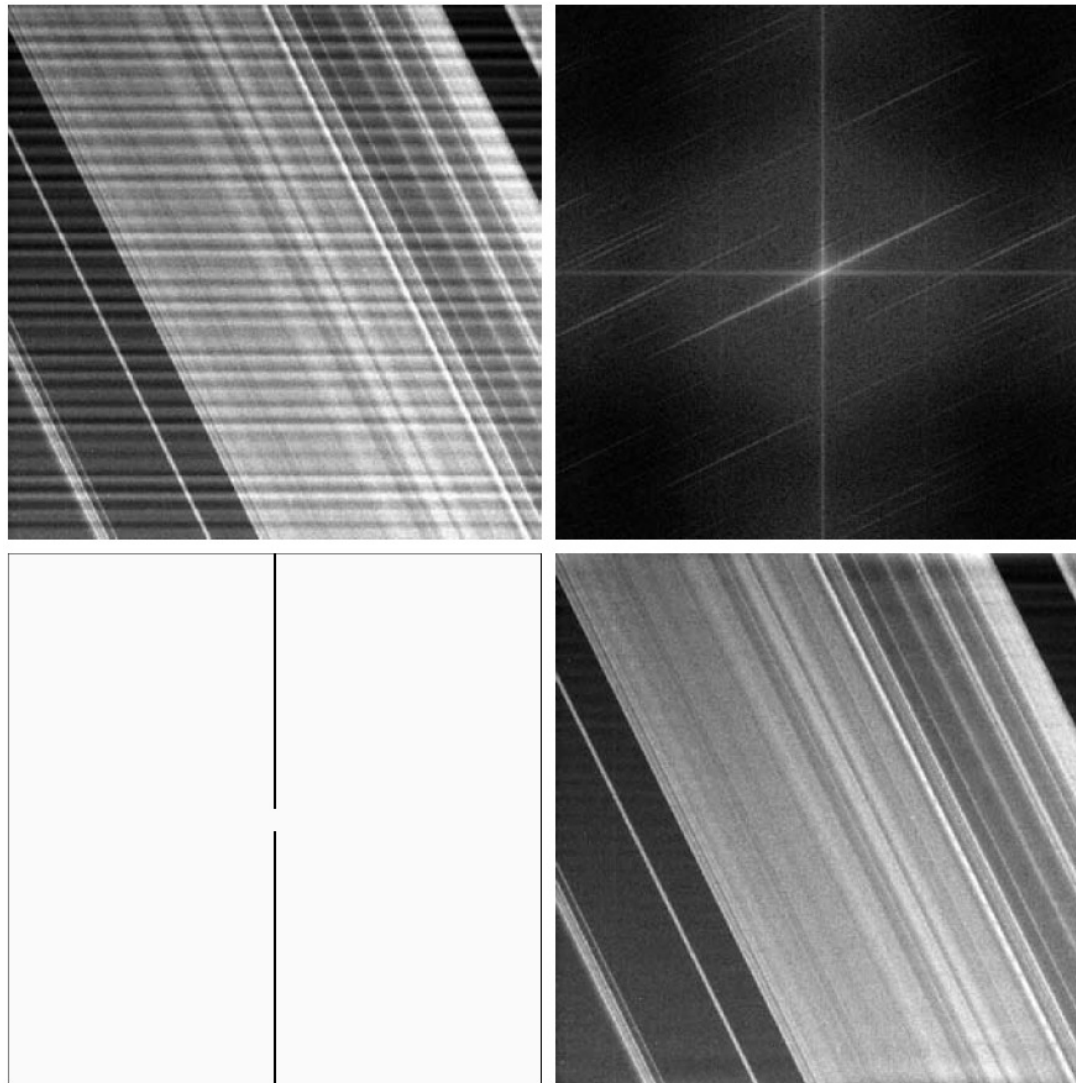
(a) Sampled newspaper image showing a moiré pattern.

(b) Spectrum.

(c) Butterworth notch reject filter multiplied by the Fourier transform.

(d) Filtered image.

Beispiel: Reduzierung Interferenzmuster

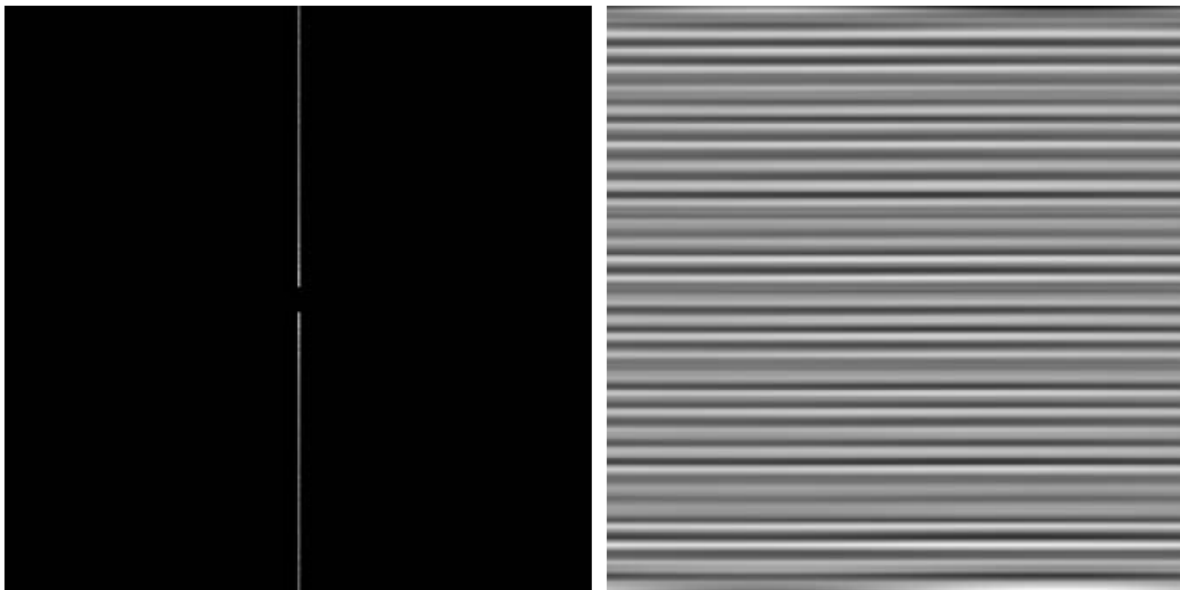


a	b
c	d

FIGURE 4.65
(a) 674×674 image of the Saturn rings showing nearly periodic interference. (b) Spectrum: The bursts of energy in the vertical axis near the origin correspond to the interference pattern. (c) A vertical notch reject filter. (d) Result of filtering. The thin black border in (c) was added for clarity; it is not part of the data. (Original image courtesy of Dr. Robert A. West, NASA/JPL.)

Beispiel: Reduzierung Interferenzmuster

- Notch-Pass-Filter isoliert Interferenzmuster



a b

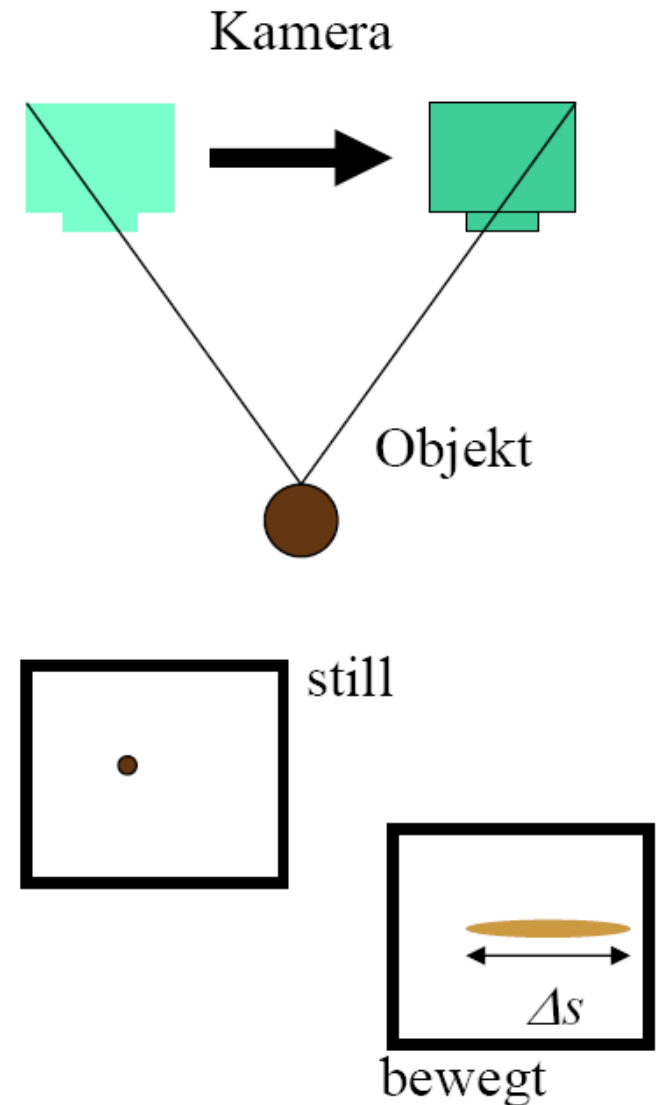
FIGURE 4.66

(a) Result (spectrum) of applying a notch pass filter to the DFT of Fig. 4.65(a).
(b) Spatial pattern obtained by computing the IDFT of (a).

RESTAURATION LINEARER BILDSTÖRUNGEN

Beispiel I: Bewegungsunschärfe

- Über einen Zeitraum Δt wird ein Objektpunkt p auf immer andere Punkte auf dem CCD-Chip abgebildet
- Bei unbewegter Kamera sei die Bildhelligkeit des abgebildeten Punkts h
- Dann ist sie bei bewegter Kamera $h/\Delta s$, wobei Δs die zurückgelegte Strecke ist
- Wenn Δs für alle Punkte gleich ist, dann lässt sich die Veränderung durch eine Faltung beschreiben



Bewegungsunschärfe

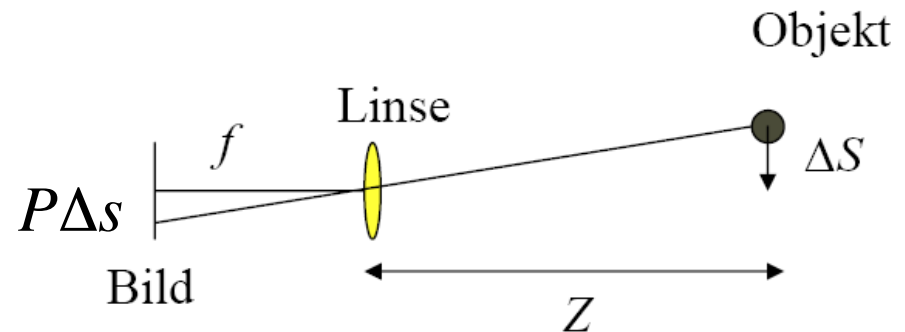
- Faltungskern ist eine Funktion w mit

$$w(t \cdot \cos \alpha, t \cdot \sin \alpha) = \begin{cases} \frac{1}{\Delta s} & |t| < \frac{\Delta s}{2} \\ 0 & \text{sonst} \end{cases}$$

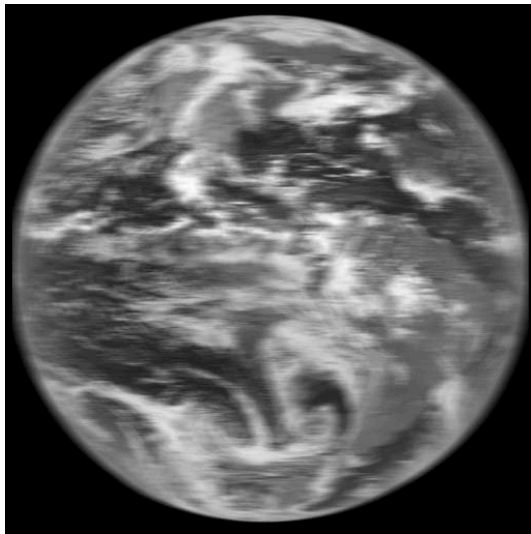
- Der Winkel α gibt die Bewegungsrichtung an
- Die Strecke $P\Delta s$ gibt die Strecke an, um die sich der Punkt bewegt hat:

$$\Delta s = \frac{f}{Z} \cdot \frac{\Delta S}{p}$$

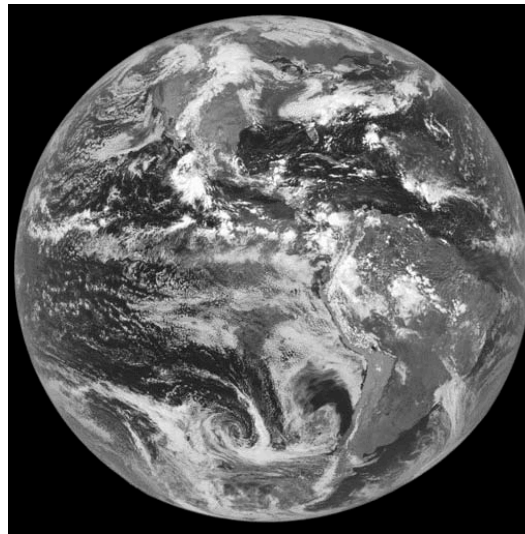
- f : Brennweite
- Z : Objektabstand
- ΔS : Bewegung in der X - Y -Ebene
- P : Pixelgröße



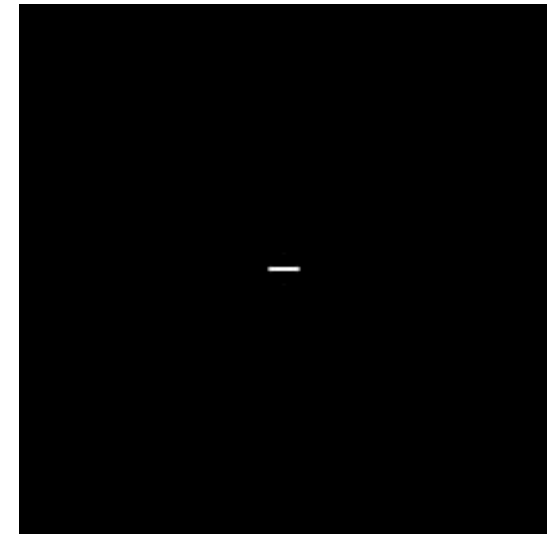
Bewegungsunschärfe



=



*



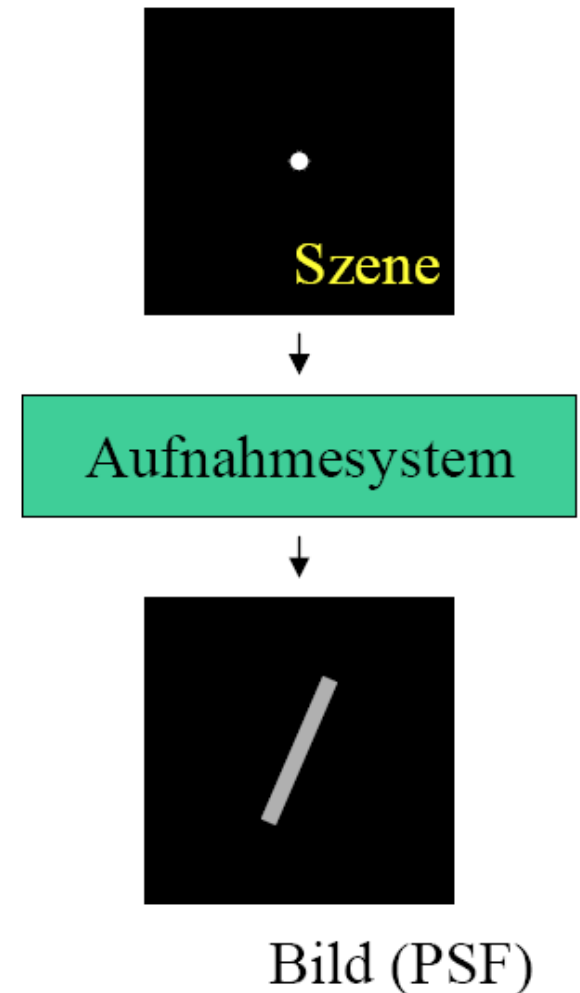
Gestörtes Bild

Original

Faltungskern

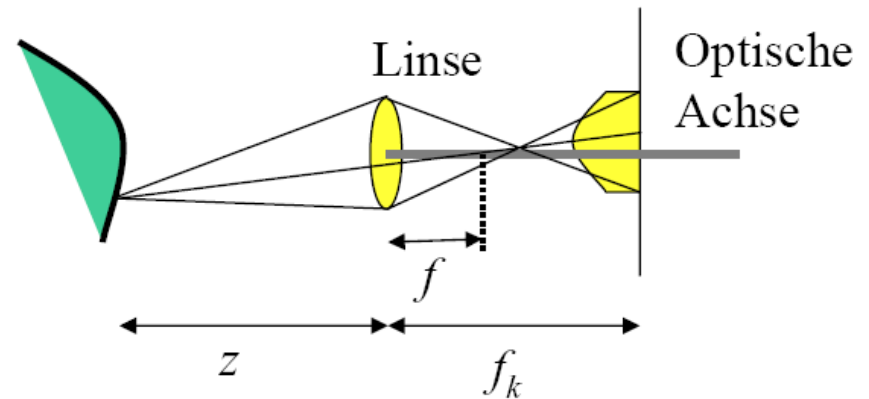
Repräsentation linearer Störungen

- Jede verschiebungsinvariante lineare Operation wird vollständig durch die Faltungsfunktion beschrieben
- Die Faltungsfunktion beschreibt die Operation für beliebige Bilder
- Die Faltungsfunktion kann als Resultat der Veränderung eines Punkts erzeugt werden
- Punktantwort = Point Spread Function (PSF)



Beispiel II: Fokussierungsunschärfe

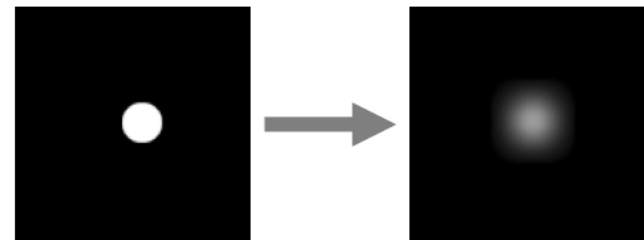
- Maß der Unschärfe hängt vom Punktabstand z , der Brennweite der Linse f und der Kammerkonstante f_k ab.



- Linsengesetz:
$$\frac{1}{z} = \frac{1}{f} + \frac{1}{f_k} \Leftrightarrow f_k = \frac{zf}{f - z}$$

- Größe des Unschärfekreises:
$$\frac{d}{f} = \frac{s}{f_k} \Leftrightarrow s = \frac{f_k d}{f} = \frac{zd}{f - z}$$

- Unschärfe kann durch Aufnahme eines punktförmigen Testobjekts angenähert werden.



Fokussierungsunschärfe



Wie kann die Störung rückgängig gemacht werden?

Bildrestauration

- Ziel: Korrektur des Bildsignals um bekannte und unbekannte Störungen
- Annahme: Störung kann durch einen verschiebungsinvarianten linearen Operator h beschrieben werden
- $g(x,y) \rightarrow$ (Störung durch PSF h) $\rightarrow g'(x,y)$
- $g'(x,y) = [h * g](x,y)$
- PSF beschreibt die Störung \rightarrow Wie kann die PSF bestimmt werden?

Invertierung der Störung

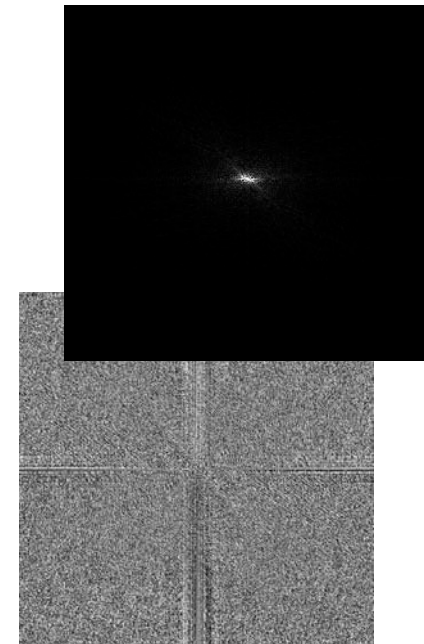
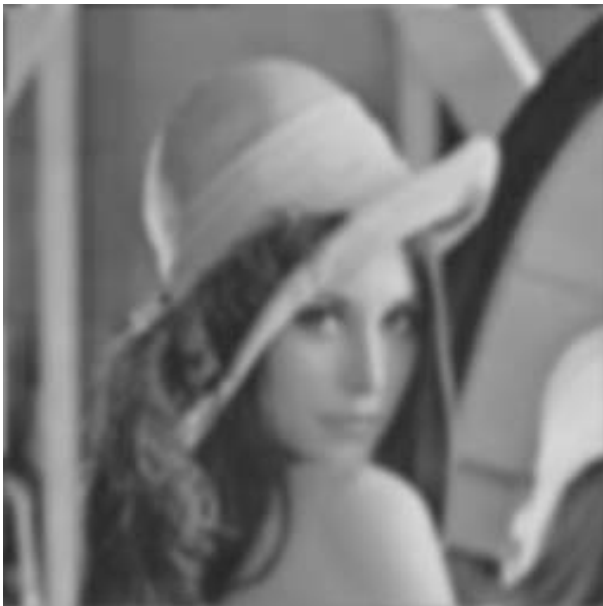
- Überführung der Repräsentation in den Frequenzraum:

$$G'(u,v) = FT[g'(m,n)] = FT[[h*g](m,n)] = H(u,v) \cdot G(u,v)$$

- Invertierung:

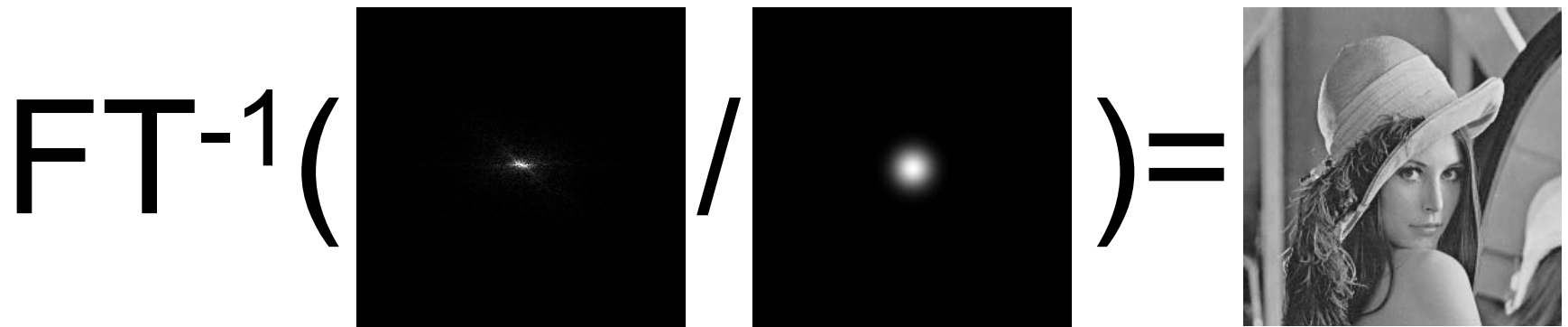
$$g(m,n) = FT^{-1}[G'(u,v)/H(u,v)]$$

← Inverse Filterung

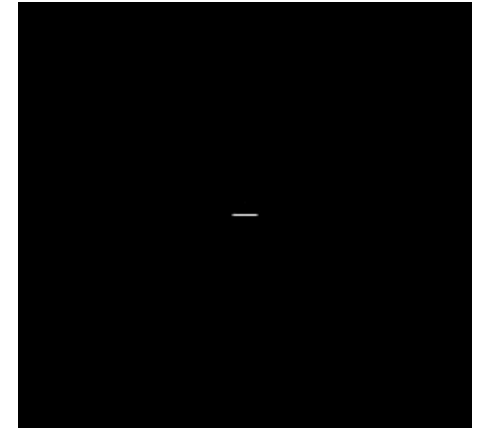


Inverse Filterung

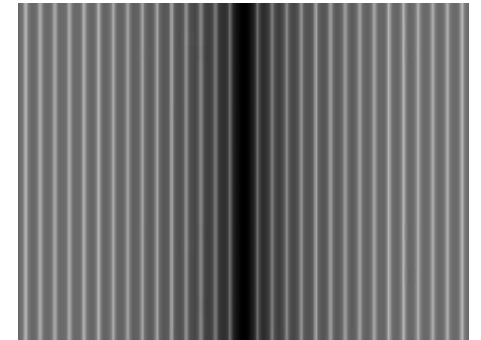
- Vollständige Rückgewinnung der Information aus den gestörten Daten

$$FT^{-1} \left(\frac{\text{[Distorted Image Spectrum]}}{\text{[Filter Spectrum]}} \right) = \text{[Original Image]}$$
The diagram illustrates the process of inverse filtering. It features the mathematical expression $FT^{-1} \left(\frac{\text{[Distorted Image Spectrum]}}{\text{[Filter Spectrum]}} \right) = \text{[Original Image]}$. The first square in the fraction represents the distorted image spectrum, showing a central peak with a surrounding halo. The second square represents the filter spectrum, which is a smooth, bell-shaped curve centered on the same point. The result of the division is the original image, a grayscale photograph of a woman wearing a hat.

Bewegungsunschärfe



PSF



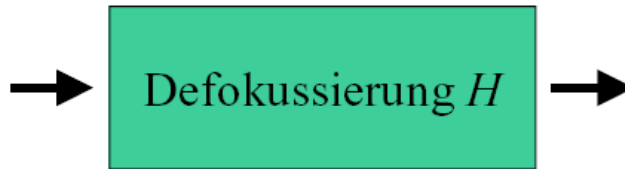
FT(PSF)

Bewegungsunschärfe

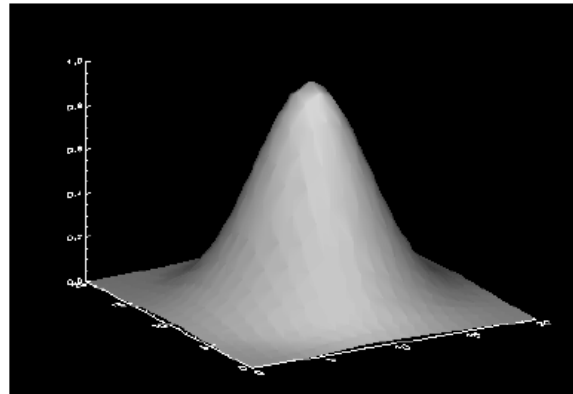
- Resultat der inversen Filterung
 $FT^{-1}[FT(g')/FT(PSF)]$



Gesucht: Point Spread Function

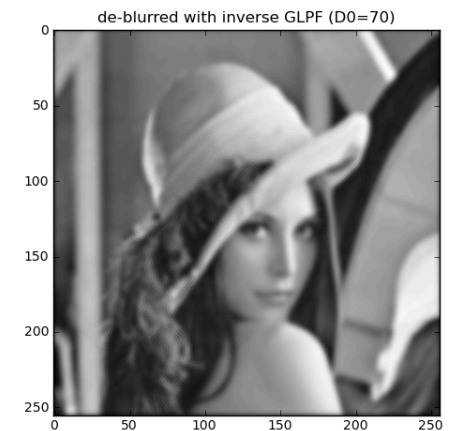
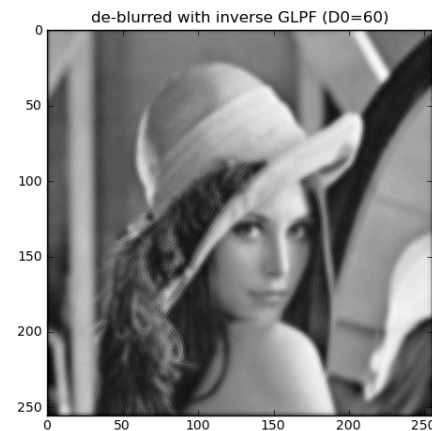
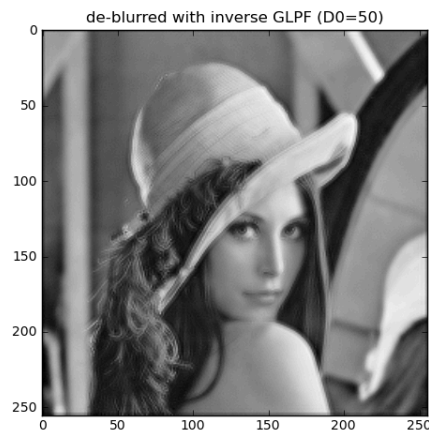
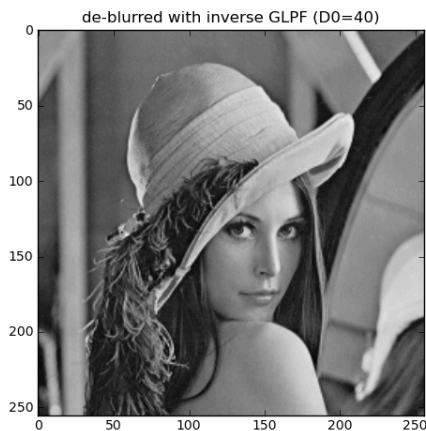
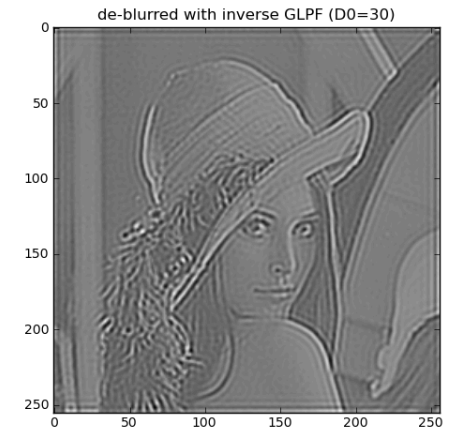
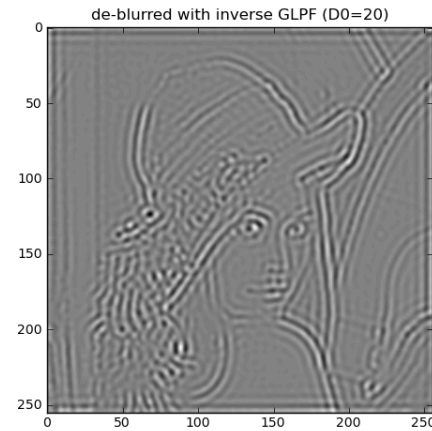
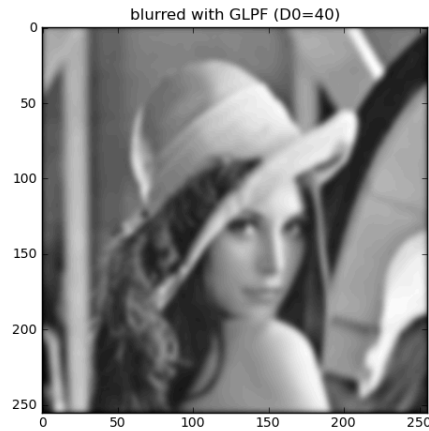
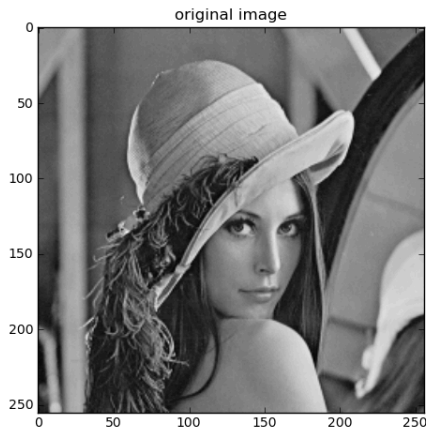


Gesucht:

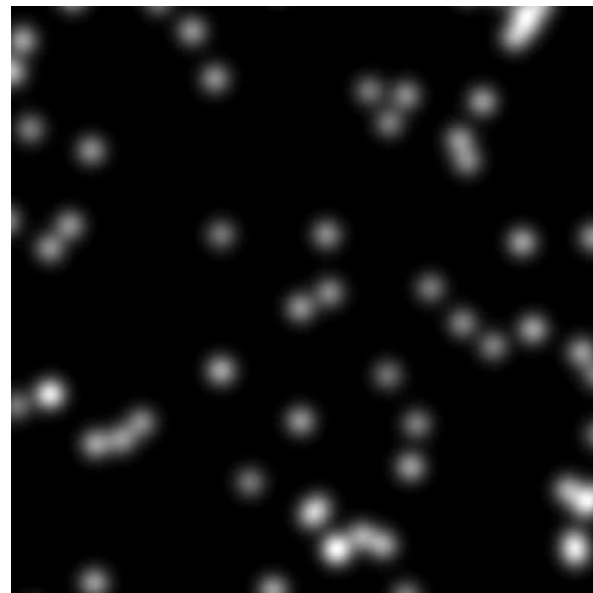


PSF

Gesucht: Point Spread Function



PSF von Testbildern



- Annahme: Störung konstant und Testaufnahme möglich
- Durch die Aufnahme eines punktförmigen Objekts kann ein δ -Impuls approximiert werden
- Aufnahme ist eine Näherung für die PSF

Numerische Probleme bei der inversen Filterung

$$g = h * f \Rightarrow f(m, n) = FT^{-1} \left(\frac{G(u, v)}{H(u, v)} \right)$$

- Problem: Nullstellen von H
 - Treten auf, falls h als Matrix nicht den vollen Rang hat
 - Auch kleine Werte von H sind numerisch schon ein Problem
- Deswegen in der Praxis:

$$F(u, v) = \begin{cases} \frac{G(u, v)}{H(u, v)} & H(u, v) > H_{\min} \\ 0 & \text{sonst} \end{cases}$$

Rauschen

- Problem: Inverse Filterung geht von idealen (ungestörten) Daten aus
- aber: Bilddaten enthalten Rauschen
- inverse Filterung verstärkt Rauschen extrem
 - mit steigender Frequenz: (weißes) Rauschen bleibt, Signal-Amplitude nimmt schnell ab, Rauschanteil wird höher

$$g(m, n) = f(m, n) * h(m, n) + \eta(m, n)$$

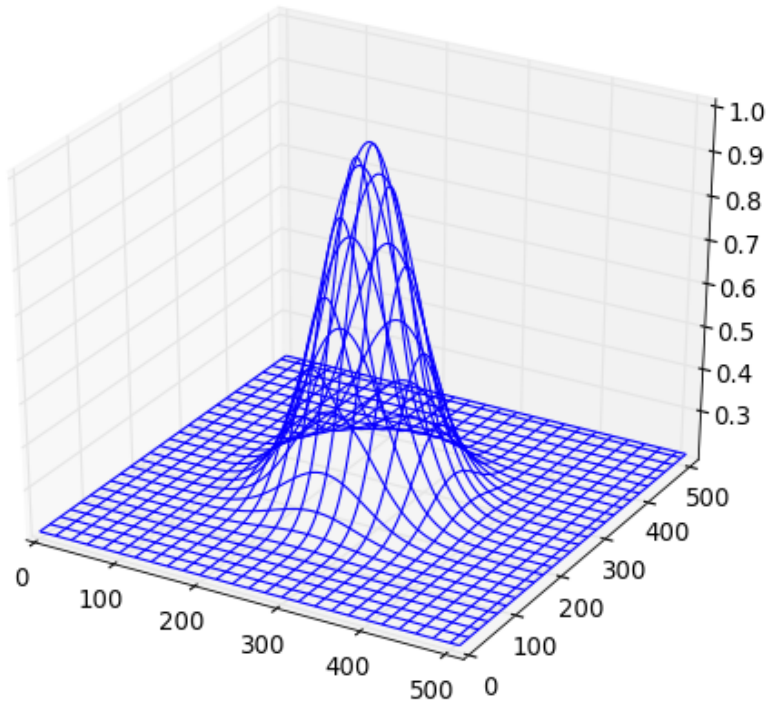
$$G(u, v) = F(u, v) \cdot H(u, v) + N(u, v) \Leftrightarrow \frac{G(u, v)}{H(u, v)} = F(u, v) + \frac{N(u, v)}{H(u, v)}$$

- ad-hoc Lösung: hohe Frequenzen ausschließen

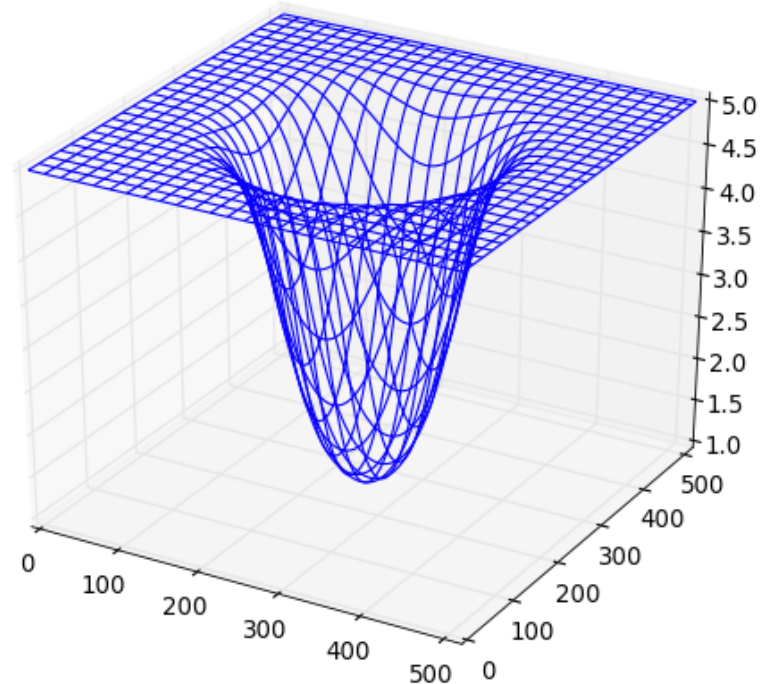
Rauschen

$$G(u, v) = F(u, v) \cdot H(u, v) + N(u, v) \Leftrightarrow \frac{G(u, v)}{H(u, v)} = F(u, v) + \frac{N(u, v)}{H(u, v)}$$

PSF: $H(u, v)$

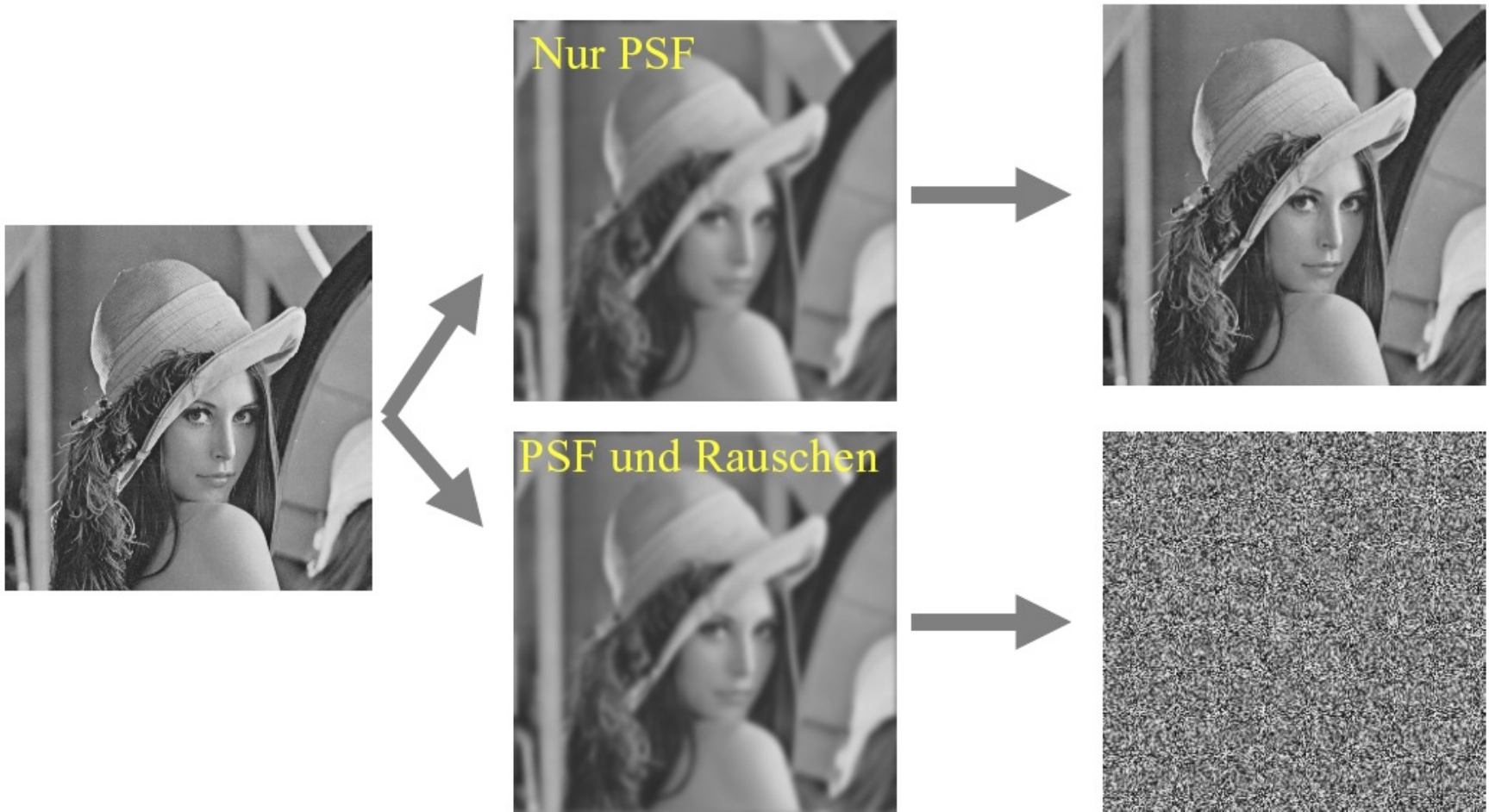


inv. PSF: $\frac{1}{H(u, v)}$



Rauschen

- Invertierung bei Rauschen oft nicht möglich

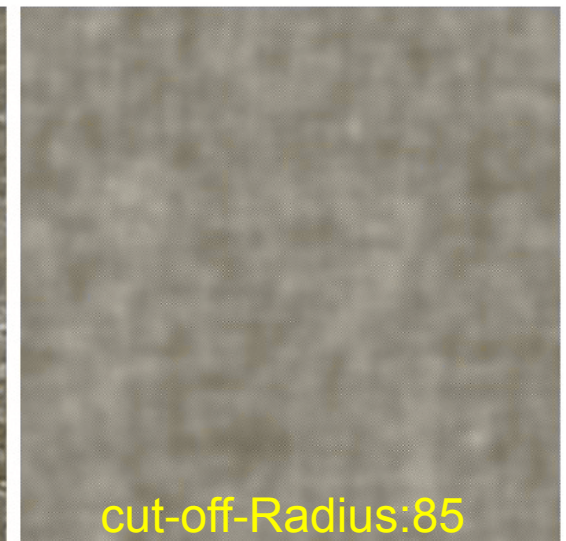
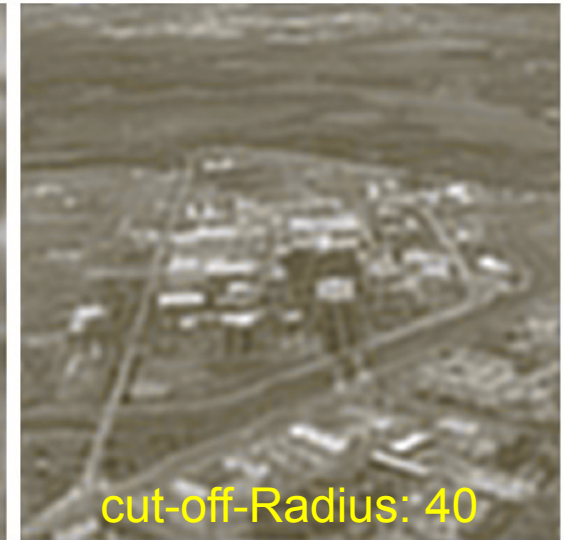
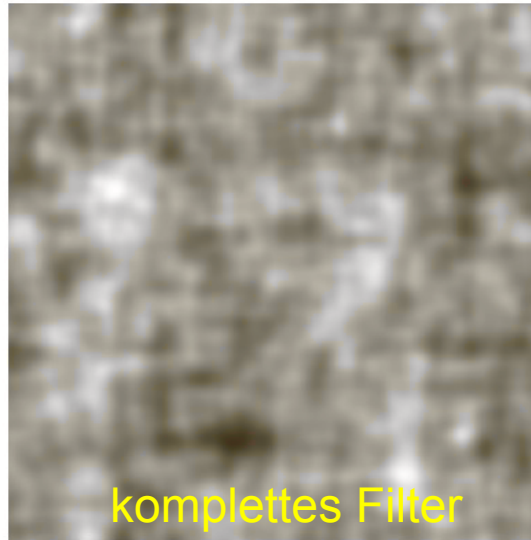


Abschneiden hoher Frequenzen

a	b
c	d

FIGURE 5.27

Restoring Fig. 5.25(b) with Eq. (5.7-1). (a) Result of using the full filter. (b) Result with H cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.



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Wiener Filter

- Minimierung des Fehlers zwischen Originalbild f und Schätzer \hat{f} führt zu

$$\hat{F}(u, v) = X(u, v) \cdot G(u, v)$$

$$X(u, v) = \frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + \frac{S_\eta(u, v)}{S_f(u, v)}} = \frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + \frac{|N(u, v)|^2}{|F(u, v)|^2}}$$

- S_η und S_f sind die Spektren (Quadrate der Amplituden) des Rauschens bzw. der ungestörten Funktion
 - $S_\eta = 0$ (ungestört) \rightarrow perfekte inverse Filterung
- Wiener Filter dämpft Frequenzen abhängig von SNR

Heuristisches Wiener Filter

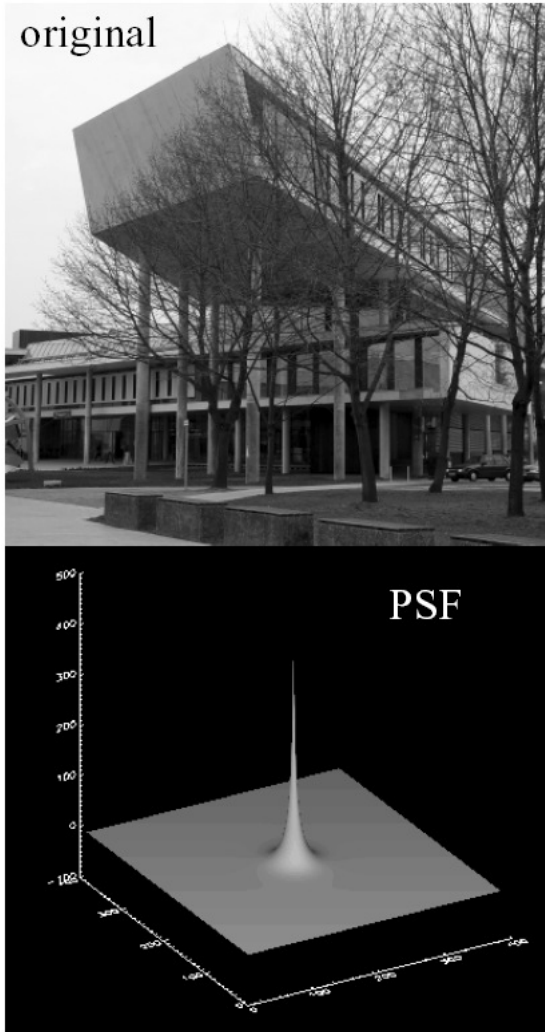
- Leider ist S_η in der Praxis meist unbekannt
- Lösung: Konstante K : heuristisches Wiener Filter

$$\hat{F}_K(u, v) = X_K(u, v) \cdot G(u, v)$$

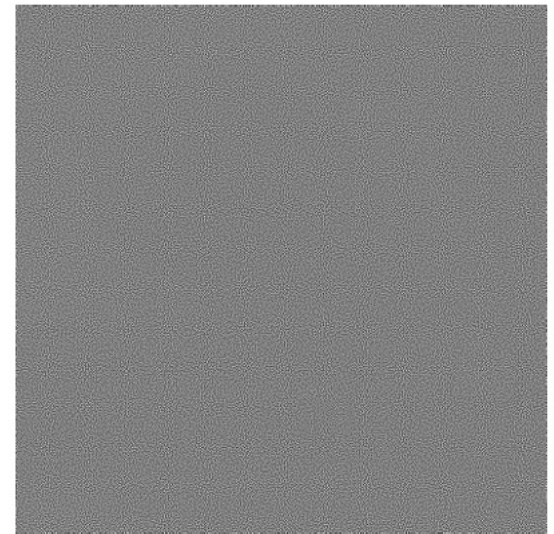
$$X_K(u, v) = \frac{1}{H(u, v)} \frac{|H(u, v)|^2}{|H(u, v)|^2 + K}$$

$$|H(u, v)|^2 = H^*(u, v)H(u, v) = \text{re}(H(u, v))^2 + \text{im}(H(u, v))^2$$

Beispiel inverse Filterung



gestörtes Bild



durch Inverse Filterung „restauriert“

Heuristisches Wiener Filter

