

# Computergrafik 2: Filtern im Frequenzraum

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Folien teilweise von Andreas Butz, sowie von Klaus D. Tönnies  
(Grundlagen der Bildverarbeitung. Pearson Studium, 2005)

# Themen heute

- Eigenschaften der Fourier-Transformation
- Filtern im Frequenzraum

# EIGENSCHAFTEN DER FOURIER-TRANSFORMATION

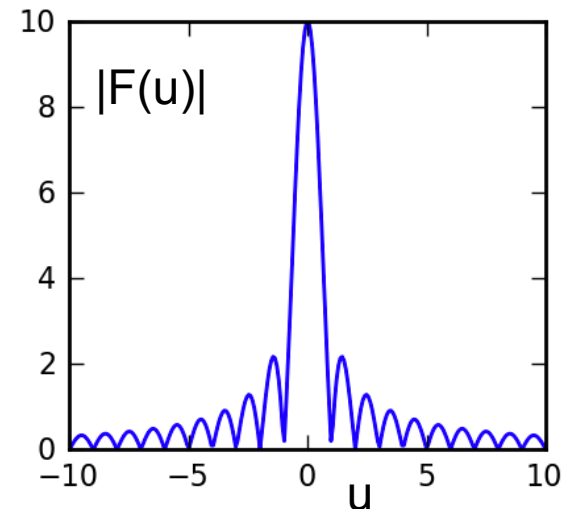
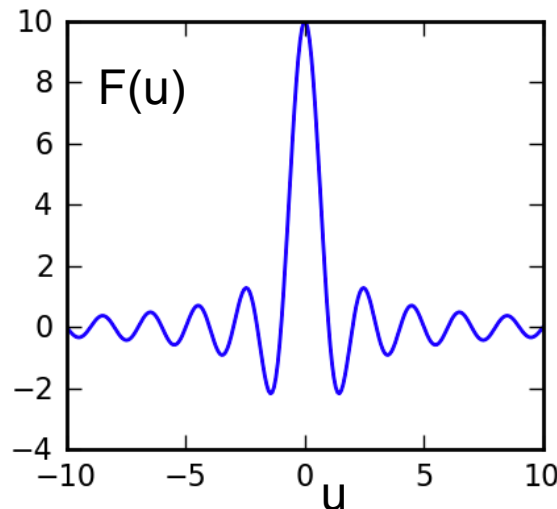
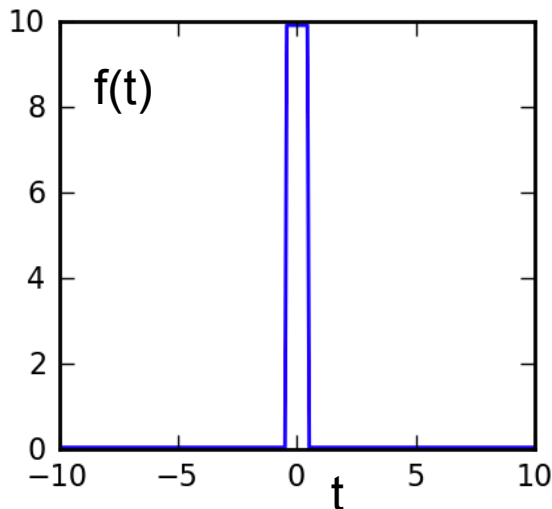
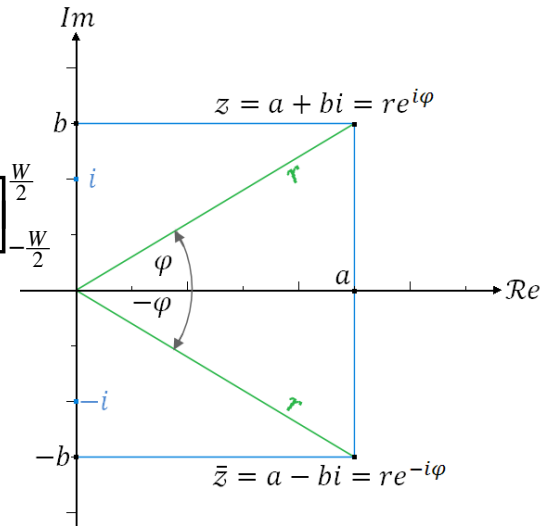
# Fourier-Transformation einer Box-Funktion

$$f(t) = \begin{cases} A & \text{falls } -\frac{W}{2} \leq t \leq \frac{W}{2} \\ 0 & \text{sonst} \end{cases}$$

$$F(u) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i2\pi ut} dt = \int_{-\frac{W}{2}}^{\frac{W}{2}} A \cdot e^{-i2\pi ut} dt = -\frac{A}{i2\pi u} \left[ e^{-i2\pi ut} \right]_{-\frac{W}{2}}^{\frac{W}{2}}$$

$$= -\frac{A}{i2\pi u} \left[ e^{-i2\pi u \frac{W}{2}} - e^{i2\pi u \frac{W}{2}} \right] = \frac{A}{i2\pi u} \left[ e^{i\pi u W} - e^{-i\pi u W} \right]$$

$$= \frac{A}{i2\pi u} i2 \sin(\pi u W) = AW \frac{\sin(\pi u W)}{\pi u W} \quad \text{“Ringing-Artefakt”}$$





# Fourier-Transformation einer Gauß-Funktion

$$f(t) = e^{-at^2}$$

$$F(u) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i2\pi ut} dt = \int_{-\infty}^{\infty} e^{-at^2} \cdot e^{-i2\pi ut} dt$$

$$= \int_{-\infty}^{\infty} e^{-at^2} \cdot [\cos(2\pi ut) - i \sin(2\pi ut)] dt$$

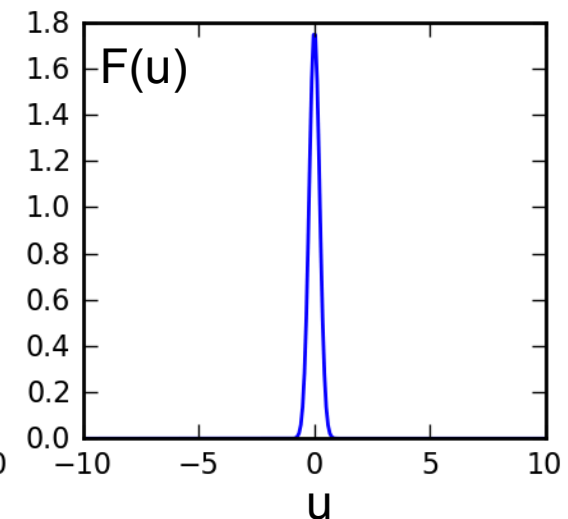
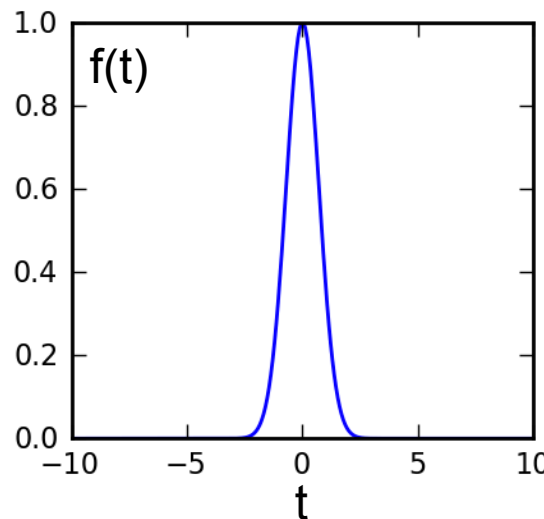
$$= \int_{-\infty}^{\infty} e^{-at^2} \cdot \cos(2\pi ut) dt - i \int_{-\infty}^{\infty} e^{-at^2} \cdot \sin(2\pi ut) dt$$

$$= \int_{-\infty}^{\infty} e^{-at^2} \cdot \cos(2\pi ut) dt - 0$$

$$= \sqrt{\frac{\pi}{a}} e^{-\pi^2 u^2 / a}$$

erstes Integral:  
Handbook of  
Mathematical  
Functions...

sin ist punktsymmetrisch  
zum Ursprung,  
insgesamt ist zweiter  
Integrand ungerade,  
symmetrische  
Integrationsgrenzen: 0

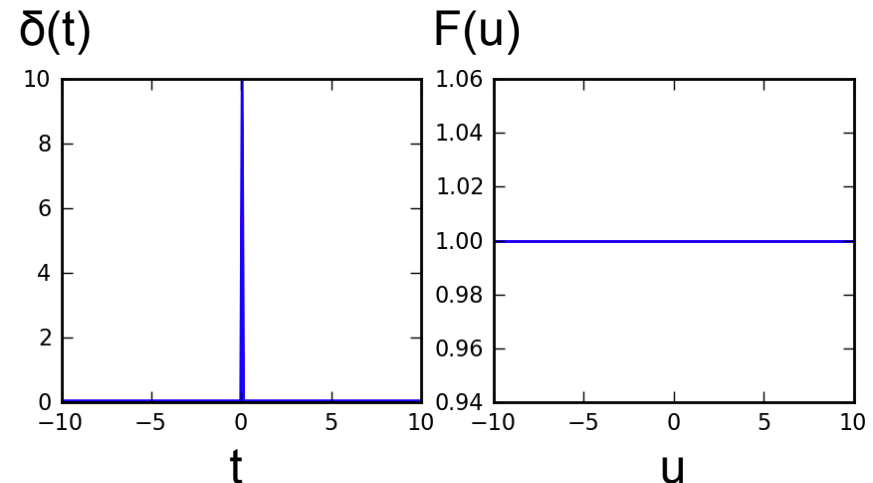


# Fourier-Transformation eines Impulses im Ortsraum

- Impuls im Ursprung wird zu Konstante im Frequenzraum

$$\delta(t) = \begin{cases} \infty & \text{falls } t = 0 \\ 0 & \text{sonst} \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$F(u) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-i2\pi ut} dt = e^{-i2\pi u \cdot 0} = e^0 = 1$$



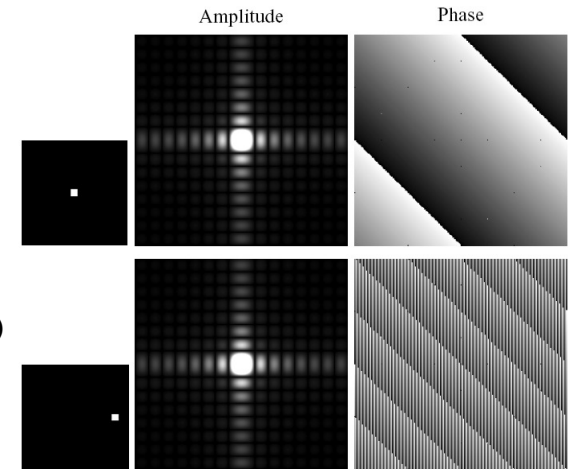
- Impuls an der Stelle  $t_0$  wird liegt auf Einheitskreis im Frequenzraum

$$F(u) = \int_{-\infty}^{\infty} \delta(t - t_0) \cdot e^{-i2\pi ut} dt = e^{-i2\pi ut_0}$$

# Eigenschaften der 2D-DFT: Translation

- 2D-DFT: 
$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-i2\pi(ux/M+vy/N)}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \cdot e^{i2\pi(ux/M+vy/N)}$$



- Verschiebung im Ortsraum führt zu Phasenverschiebung im Frequenzraum:

$$f(x - x_0, y - y_0) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \cdot e^{i2\pi(u(x-x_0)/M+v(y-y_0)/N)}$$

$$= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \cdot e^{i2\pi(ux/M+vy/N)} \cdot e^{-i2\pi(ux_0/M+vy_0/N)}$$

$$= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \left[ F(u, v) \cdot e^{-i2\pi(ux_0/M+vy_0/N)} \right] \cdot e^{i2\pi(ux/M+vy/N)}$$

# Eigenschaften der 2D-DFT: Periodizität

- 2D-DFT 
$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-i2\pi(ux/M+vy/N)}$$

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \cdot e^{i2\pi(ux/M+vy/N)}$$

- Verschiebung im Ortsraum um Vielfache von M, N:

$$f(x - aM, y - bN) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \left[ F(u, v) \cdot e^{-i2\pi(u(aM)/M+v(bN)/N)} \right] \cdot e^{i2\pi(ux/M+vy/N)}$$

$$= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \left[ F(u, v) \cdot e^{-i2\pi(ua+vb)} \right] \cdot e^{i2\pi(ux/M+vy/N)}, \quad u, v \in N_0, a, b \in \mathbb{Z}$$

$$= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) \cdot e^{i2\pi(ux/M+vy/N)} = f(x, y)$$

# Zentrierung der 2D-DFT

- Verschiebung im Frequenzraum um  $M/2, N/2$
- Visualisierung der verschobenen Funktion vorteilhaft

$$f(x, y)(-1)^{x+y} \Leftrightarrow F\left(u - \frac{M}{2}, v - \frac{N}{2}\right)$$

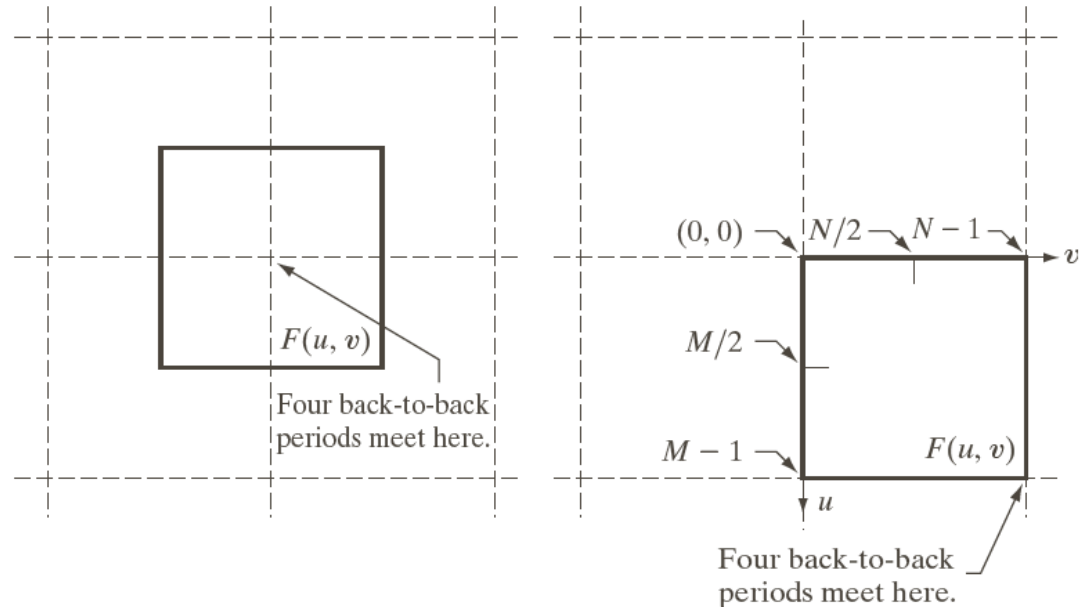
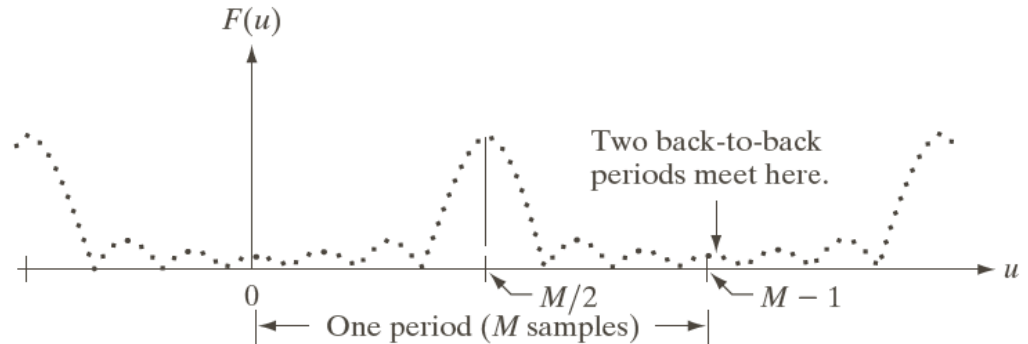
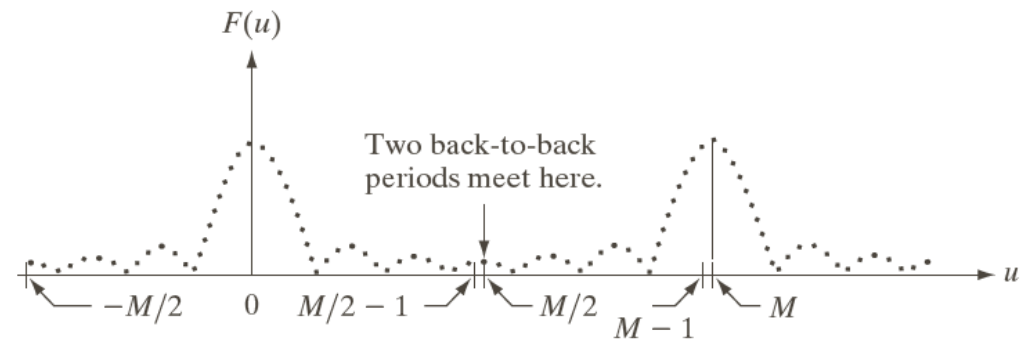


Abbildung: © R. C. Gonzalez & R. E. Woods, Digital Image Processing

# Zentrierung der 2D-DFT

- Verschiebung um  $M/2$ ,  $N/2$  im Frequenzraum:

$$F\left(u - \frac{M}{2}, v - \frac{N}{2}\right) \Leftrightarrow f(x, y)(-1)^{x+y}$$

$$F\left(u - \frac{M}{2}, v - \frac{N}{2}\right) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{-i2\pi\left(\left(u - \frac{M}{2}\right)x/M + \left(v - \frac{N}{2}\right)y/N\right)}$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \cdot e^{i\pi(x+y)} \cdot e^{-i2\pi(ux/M + vy/N)}$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[ f(x, y) \cdot (-1)^{x+y} \right] \cdot e^{-i2\pi(ux/M + vy/N)}$$

# Eigenschaften der 2D-FT: Rotation

- Rotation im Ortsraum führt zu Rotation im Frequenzraum
- Schreibe FT in Matrix-Vektor-Notation:

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) \cdot e^{i2\pi(xu+yv)} du dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot e^{i2\pi\left(\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} u \\ v \end{pmatrix}\right)} du dv, \quad \text{mit } \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} u \\ v \end{pmatrix} = xu + yv$$



- Rotation mit 2x2-Rotationsmatrix R:

$$f\left(R\begin{pmatrix} x \\ y \end{pmatrix}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot e^{i2\pi\left(R\begin{pmatrix} x \\ y \end{pmatrix}\right)^T \begin{pmatrix} u \\ v \end{pmatrix}} du dv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(R\begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot e^{i2\pi(xu+yv)} du dv$$

nächste Folie

# Eigenschaften der 2D-FT: Rotation

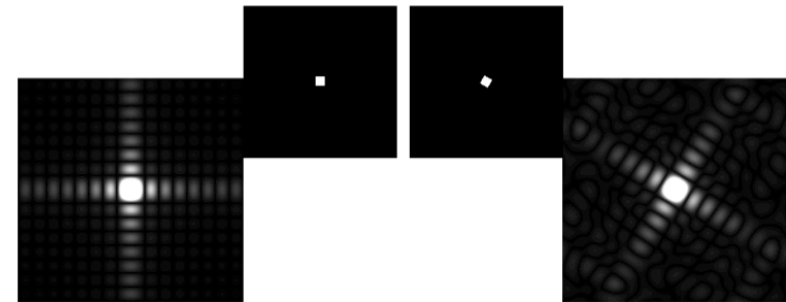
$$f\left(R\begin{pmatrix} x \\ y \end{pmatrix}\right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot e^{i2\pi\left(\left(R\begin{pmatrix} x \\ y \end{pmatrix}\right)^T \begin{pmatrix} u \\ v \end{pmatrix}\right)} du dv \quad \text{Rotationsmatrix } R$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(R\begin{pmatrix} u' \\ v' \end{pmatrix}\right) \cdot e^{i2\pi\left(\left(R\begin{pmatrix} x \\ y \end{pmatrix}\right)^T R\begin{pmatrix} u' \\ v' \end{pmatrix}\right)} du dv, \quad \text{mit } \begin{pmatrix} u' \\ v' \end{pmatrix} = R^{-1}\begin{pmatrix} u \\ v \end{pmatrix}, \quad R^{-1} = R^T$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(R\begin{pmatrix} u' \\ v' \end{pmatrix}\right) \cdot e^{i2\pi\left(\begin{pmatrix} x \\ y \end{pmatrix}^T R^T R\begin{pmatrix} u' \\ v' \end{pmatrix}\right)} du dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(R\begin{pmatrix} u' \\ v' \end{pmatrix}\right) \cdot e^{i2\pi\left(\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} u' \\ v' \end{pmatrix}\right)} du dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) \cdot e^{i2\pi(xu+yv)} du dv$$



Integration über alle  $(u,v)^T$ , d.h. über die gesamte Ebene, und damit auch über alle  $R^{-1}(u,v)^T$ . Substituierung  $R^{-1}(u,v)^T \rightarrow (u,v)^T$



# Konvolution im Frequenzraum

- 1D-Konvolution

$$f(x) * h(x) = \sum_{n=0}^{N-1} f(n) \cdot h(x - n)$$

$$\Leftrightarrow F(u) \cdot H(u)$$

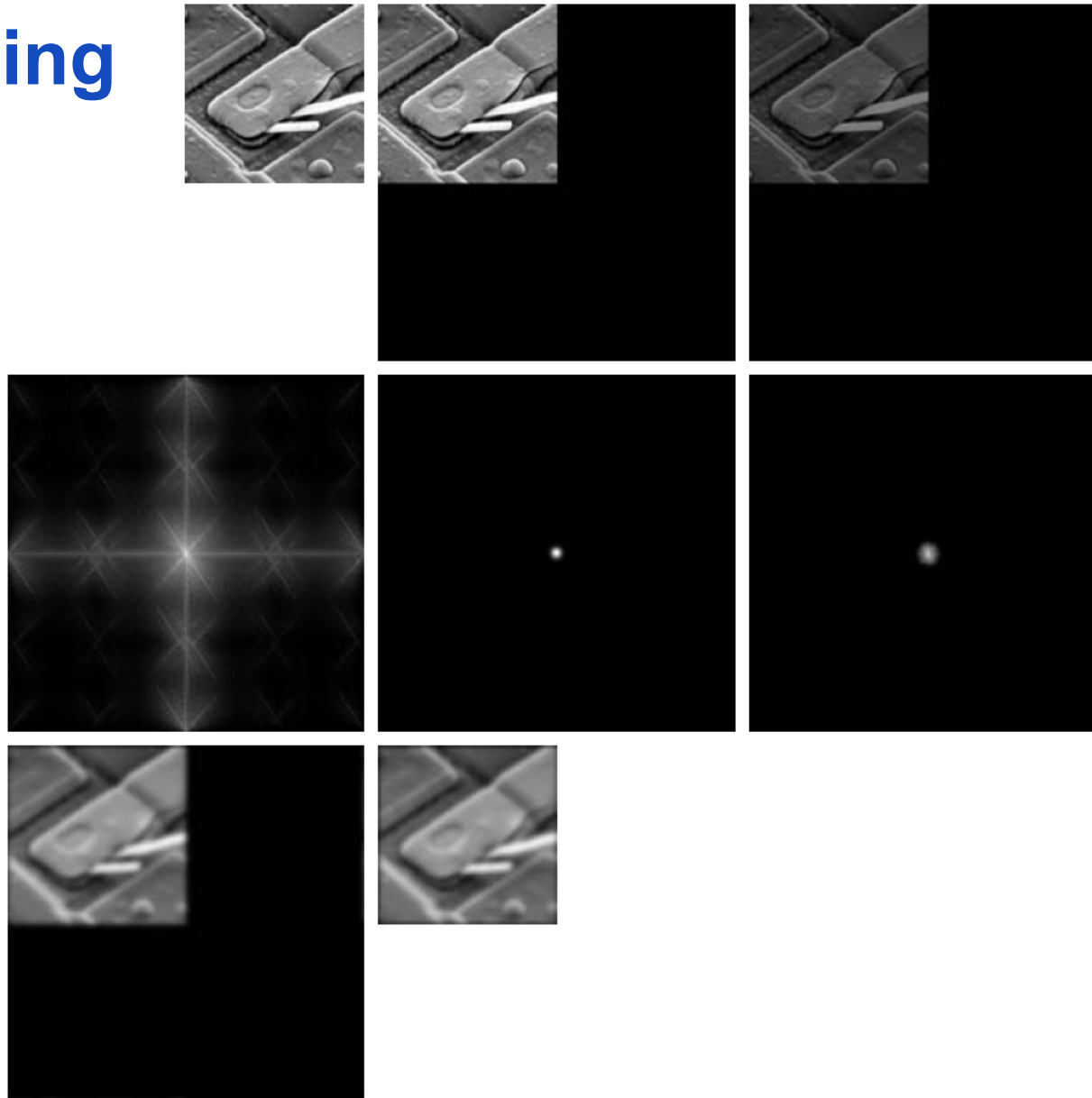
- 2D-Konvolution

$$f(x, y) * h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) \cdot h(x - m, y - n)$$

$$\Leftrightarrow F(u, v) \cdot H(u, v)$$

- Periodizität problematisch  $\rightarrow$  Padding

# Padding



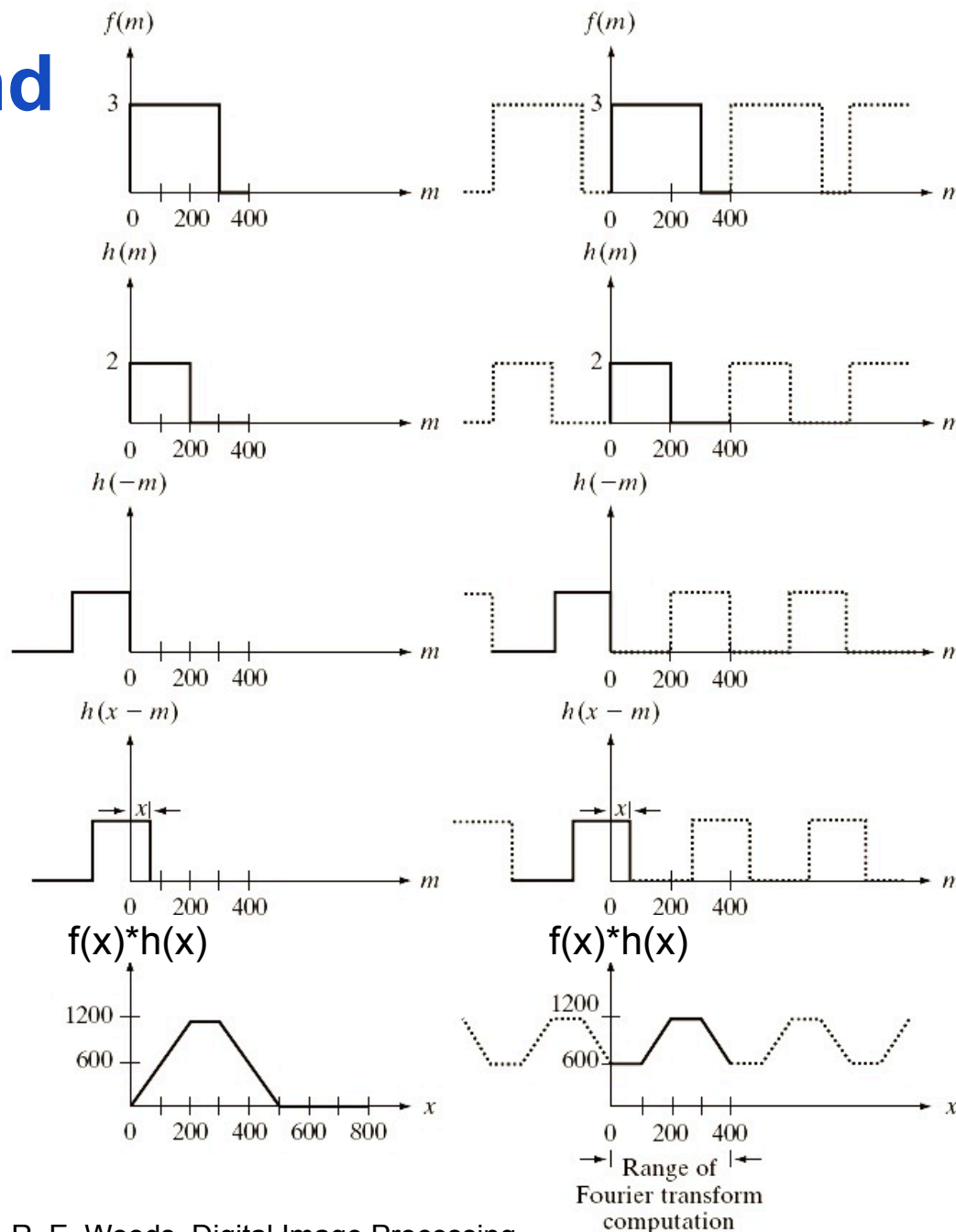
a	b	c
d	e	f
g	h	

**FIGURE 4.36**  
 (a) An  $M \times N$  image,  $f$ .  
 (b) Padded image,  $f_p$  of size  $P \times Q$ .  
 (c) Result of multiplying  $f_p$  by  $(-1)^{x+y}$ .  
 (d) Spectrum of  $F_p$ . (e) Centered Gaussian lowpass filter,  $H$ , of size  $P \times Q$ .  
 (f) Spectrum of the product  $HF_p$ .  
 (g)  $g_p$ , the product of  $(-1)^{x+y}$  and the real part of the IDFT of  $HF_p$ .  
 (h) Final result,  $g$ , obtained by cropping the first  $M$  rows and  $N$  columns of  $g_p$ .

# Wraparound Error

- Konvolution periodischer Funktionen
- Padding:  
 $P \geq A + B - 1$

$$f(x) * h(x) = \sum_{m=0}^{399} f(m) \cdot h(x - m)$$



a	f
b	g
c	h
d	i
e	j

**FIGURE 4.28** Left column: convolution of two discrete functions obtained using the approach discussed in Section 3.4.2. The result in (e) is correct. Right column: Convolution of the same functions, but taking into account the periodicity implied by the DFT. Note in (j) how data from adjacent periods produce wraparound error, yielding an incorrect convolution result. To obtain the correct result, function padding must be used.

# **FILTERN IM FREQUENZRAUM**

# Frequenzraumfilterung

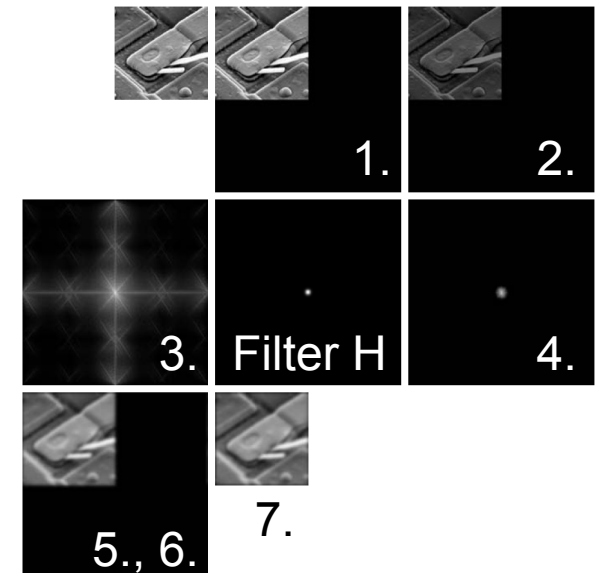
- Statt Konvolution im Ortsraum nun Multiplikation im Frequenzraum

$$f(x, y) * h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) \cdot h(x - m, y - n)$$

$$\Leftrightarrow F(u, v) \cdot H(u, v)$$

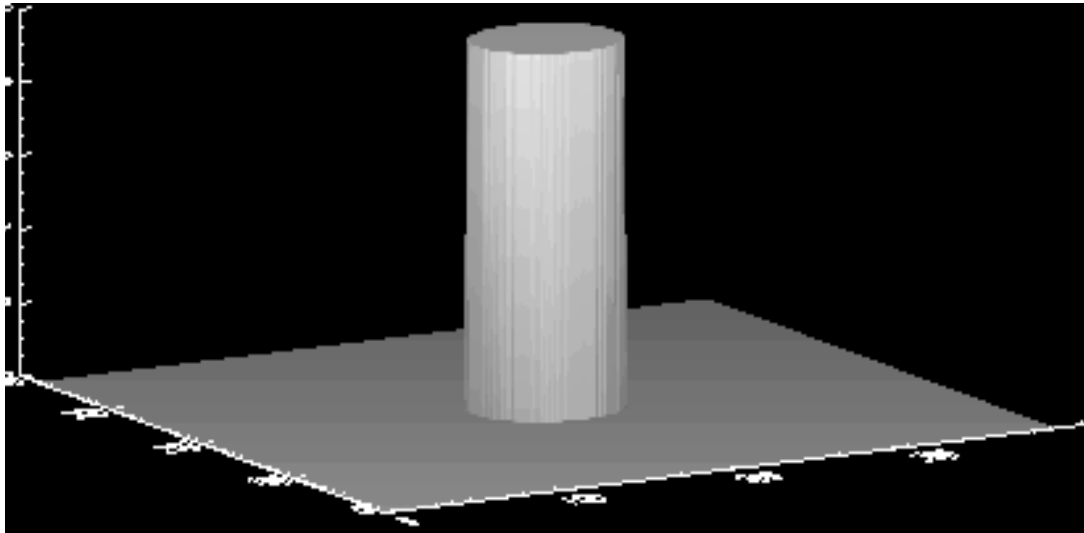
- Vorgehen

1. Padding im Ortsraum auf  $2M, 2N$
2. Multiplikation mit  $(-1)^{x+y}$
3. DFT
4. Multiplikation mit Filter im Frequenzraum
5. inverse DFT, Realteil
6. Multiplikation mit  $(-1)^{x+y}$
7. Padding entfernen



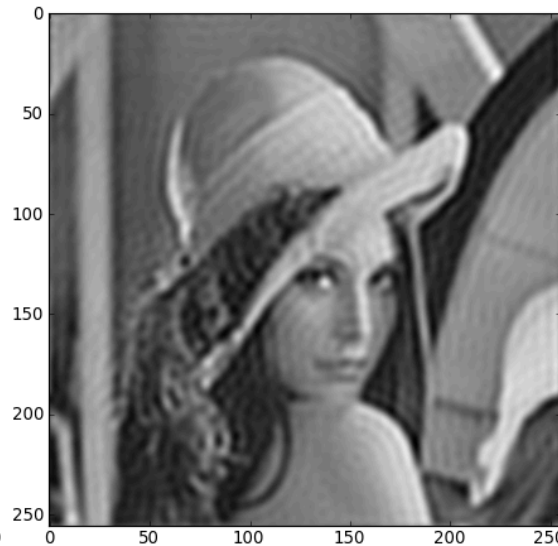
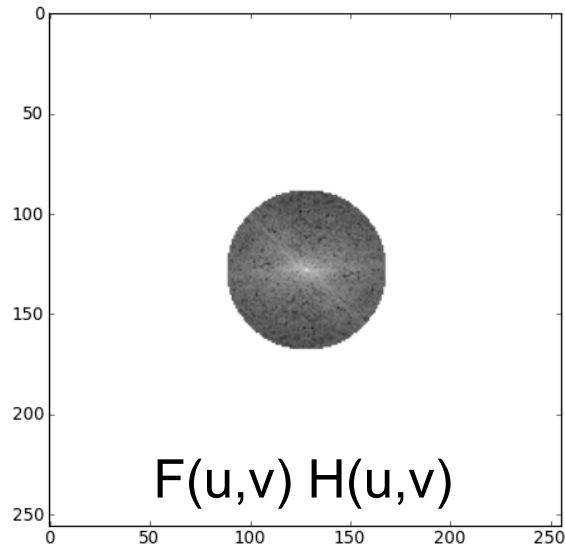
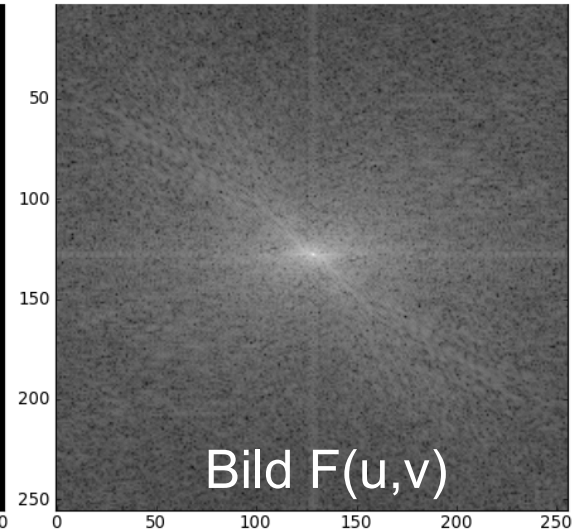
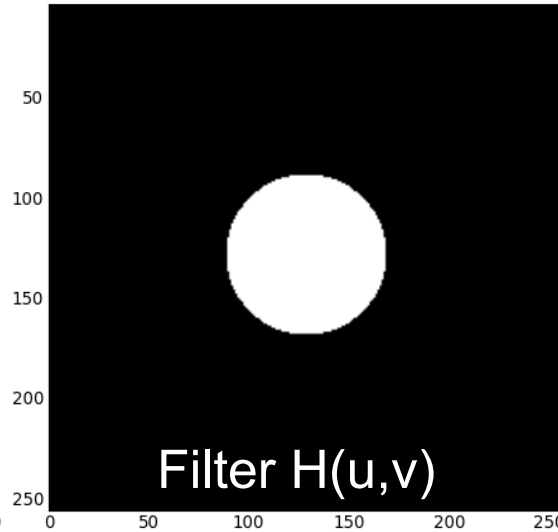
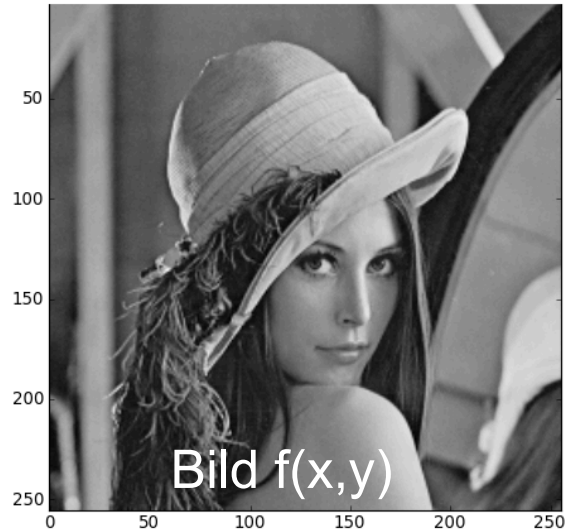
# Ideales Tiefpassfilter

- Tiefpassfilter lässt tiefe Frequenzen passieren und dämpft hohe Frequenzen
- Ideales Tiefpassfilter  $H_{F_{\max}}(u, v) = \begin{cases} 1 & , \text{ falls } u^2 + v^2 \leq F_{\max}^2 \\ 0 & , \text{ sonst.} \end{cases}$ 
  - ideal low pass filter (ILPF)
- $F_{\max}$ : Cut-Off-Frequenz



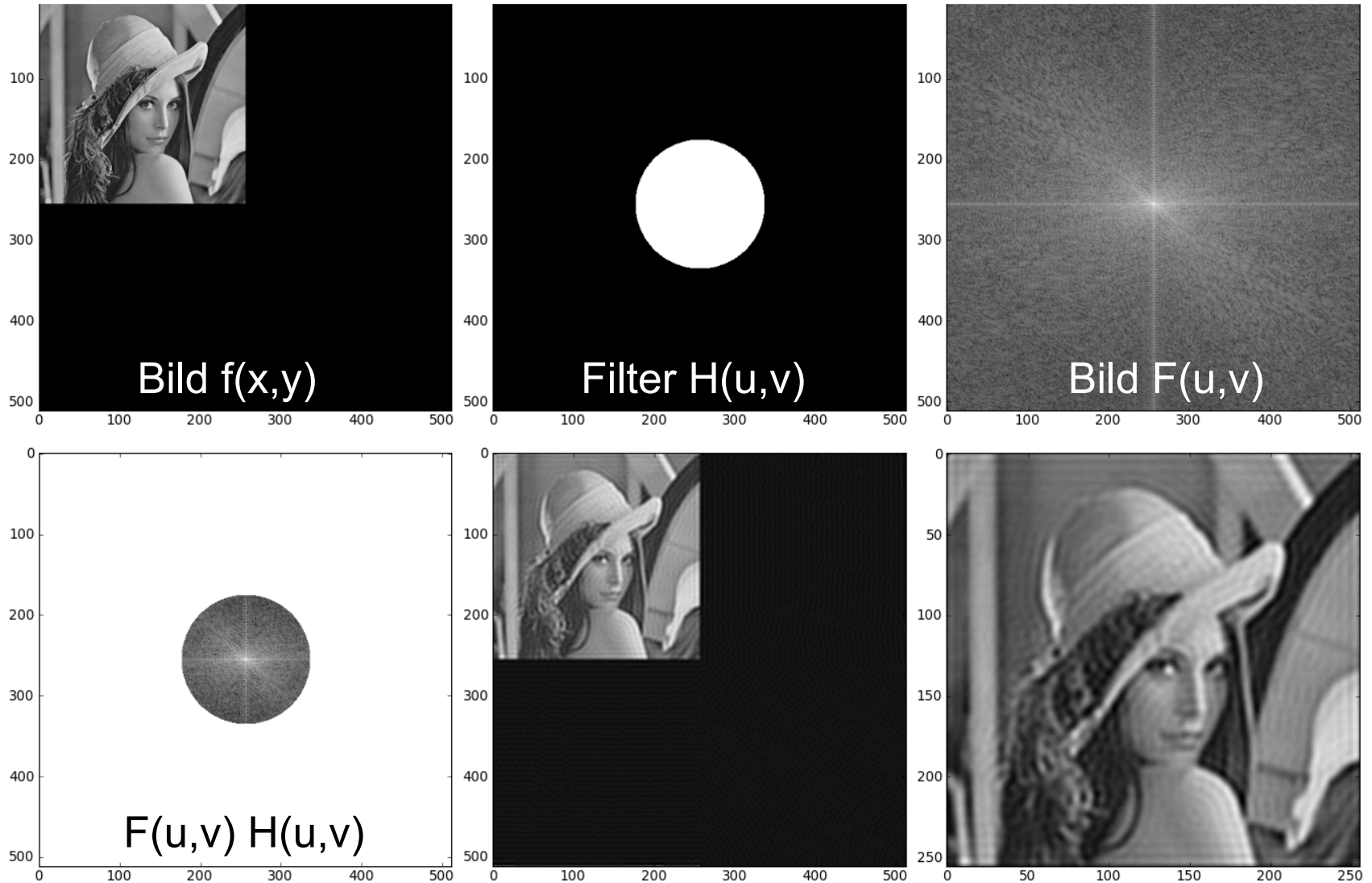
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# Ideales Tiefpassfilter (ohne Padding)



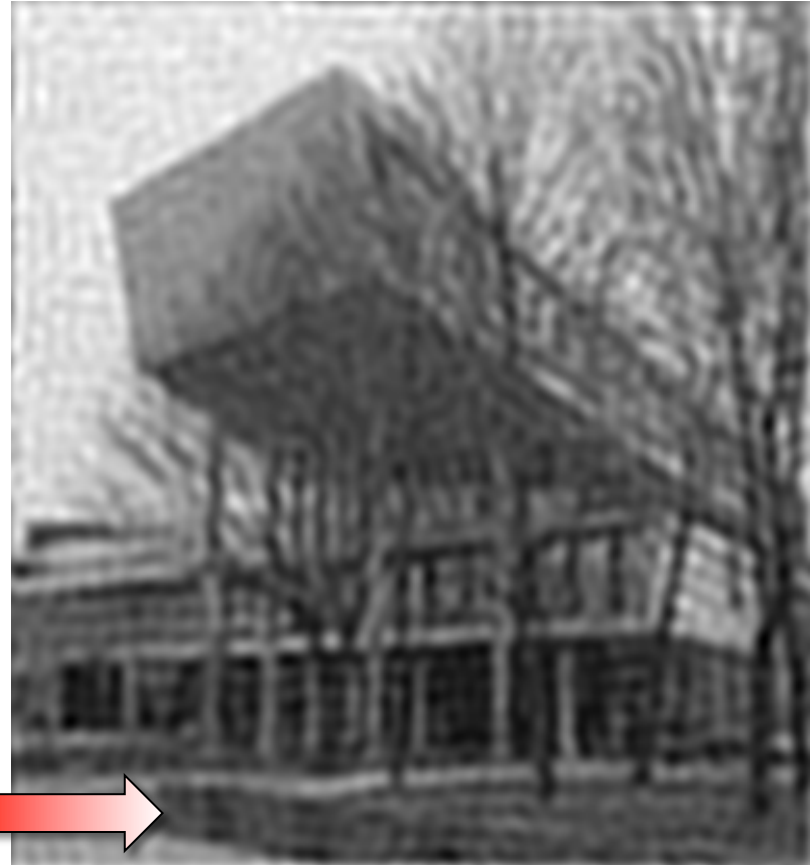
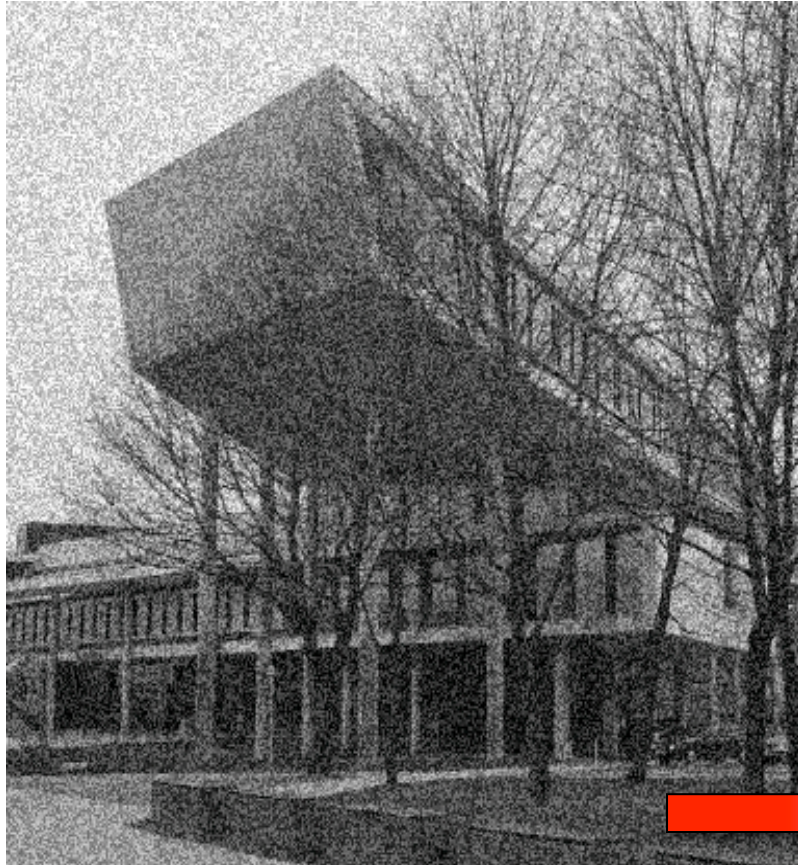
- Bild: 256x256 Pixel
- Filterradius (Cut-Off-Frequenz): 40 Pixel

# Ideales Tiefpassfilter (mit Padding)





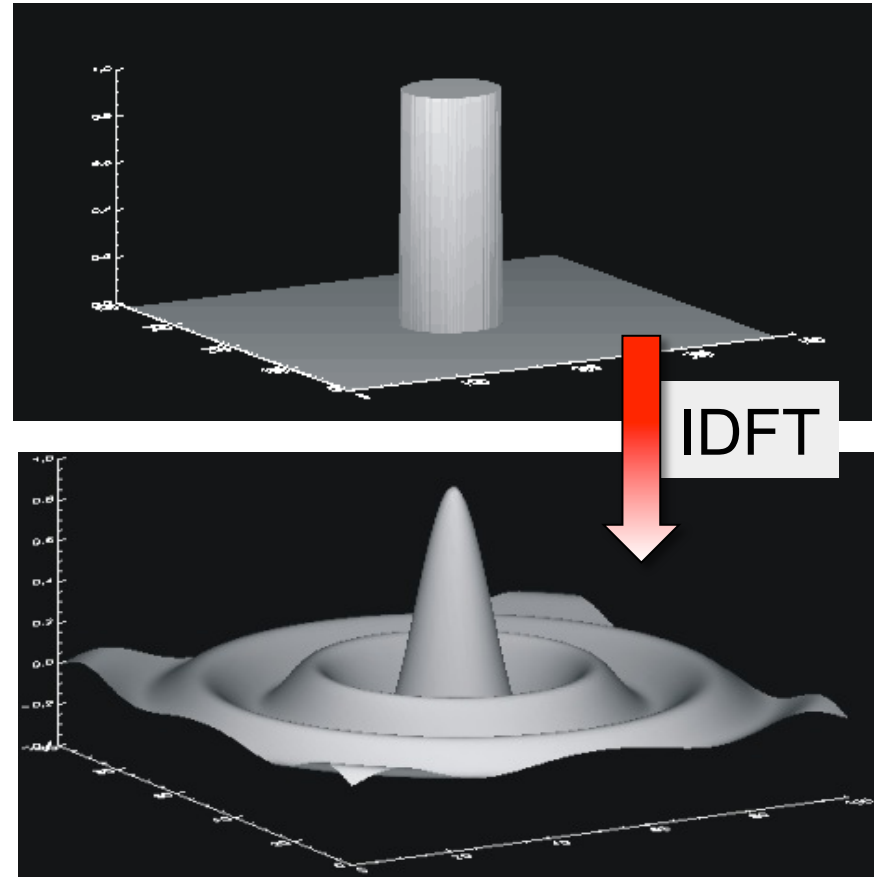
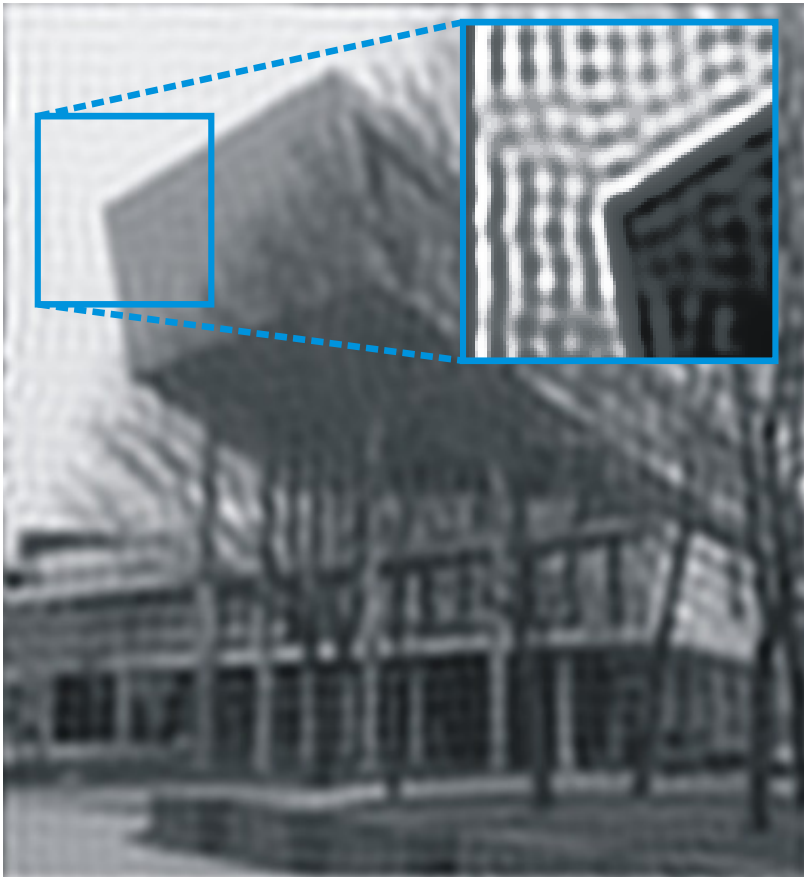
# Ideales Tiefpassfilter zur Rauschunterdrückung



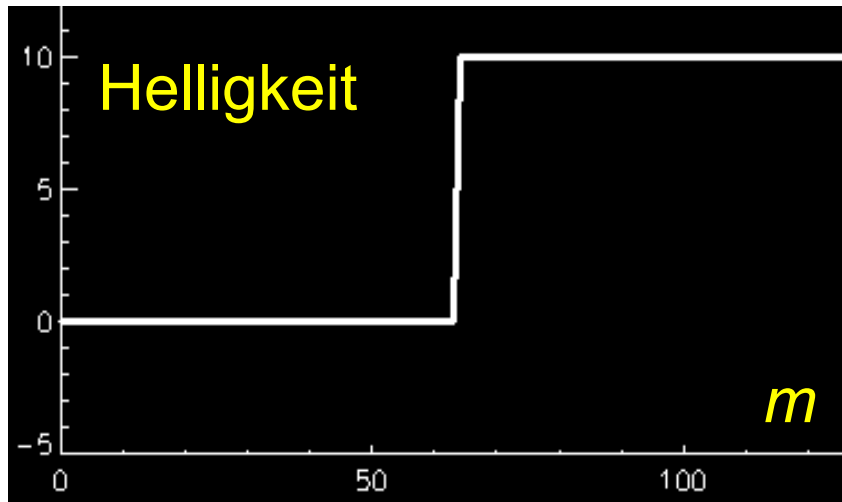
Cut-Off-Frequenz: 40

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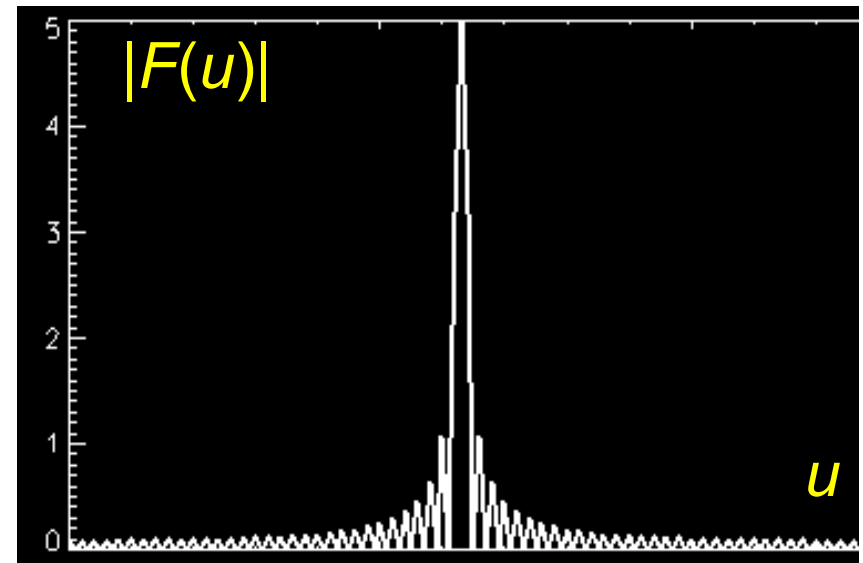
# Ringing-Artefakt



# Ringling-Artefakt



Bildzeile

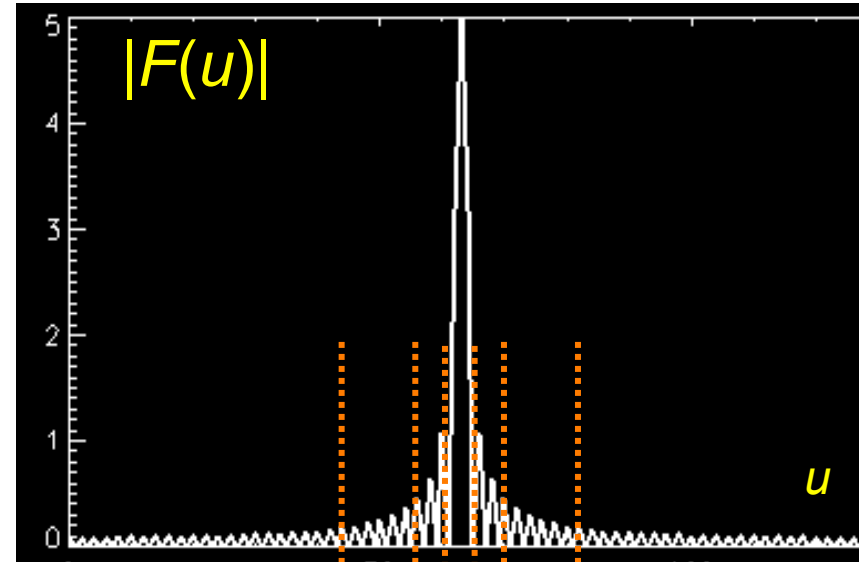
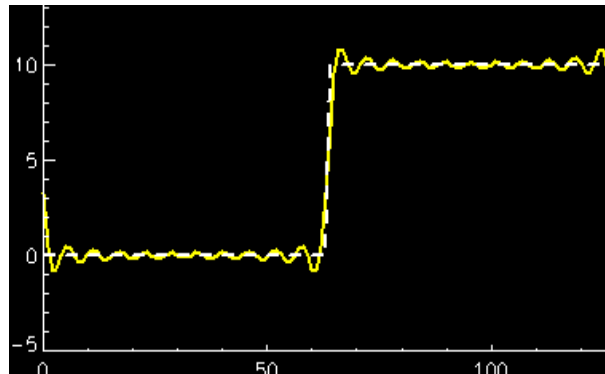
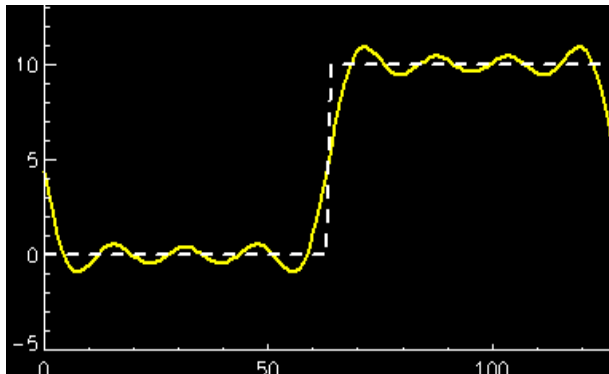
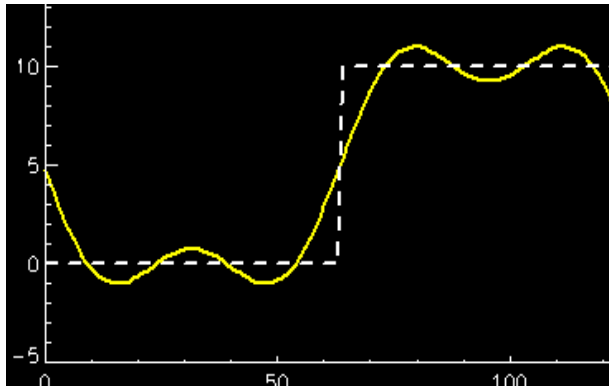


Fourier-transformierte Zeile

Das Ringling-Artefakt entsteht, weil scharfe Kanten durch Wellen **aller** Frequenzen beschrieben werden und der ideale LPF hohe Frequenzen abschneidet

# Ringling-Artefakt

je kleiner die Cut-Off-Frequenz  $F_{\max}$ , desto stärker der Ringling-Artefakt



IDFT

IDFT

IDFT

$F_{\max}$

invers Fourier-transformierte Zeile

# Butterworth-Filter

- Frequenzen werden nicht gelöscht, sondern nur abgeschwächt

- Tiefpass-Filter (BLPF):

$$H(u, v) = \frac{1}{1 + (D(u, v) / D_0)^{2n}}$$

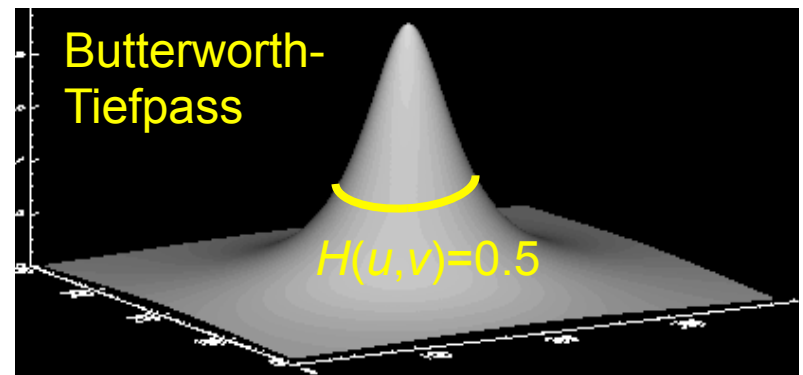
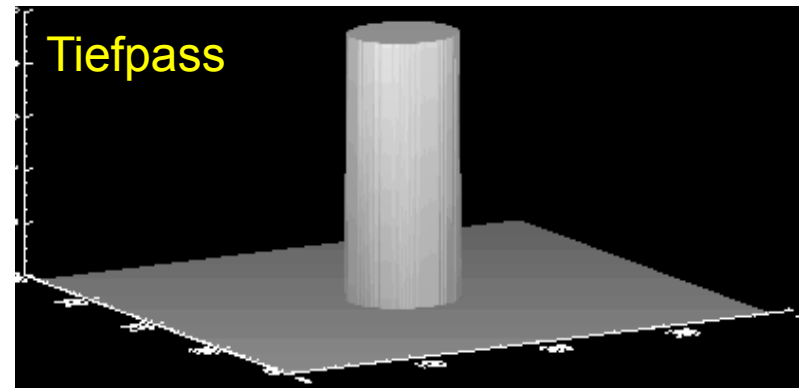
- Hochpass-Filter (BHPF):

$$H(u, v) = \frac{1}{1 + (D_0 / D(u, v))^{2n}}$$

$D_0$ : Cutoff-Frequenz

$D(u, v)$ : Frequenz, Abst. Ursprung

$n$ : Ordnung des Filters



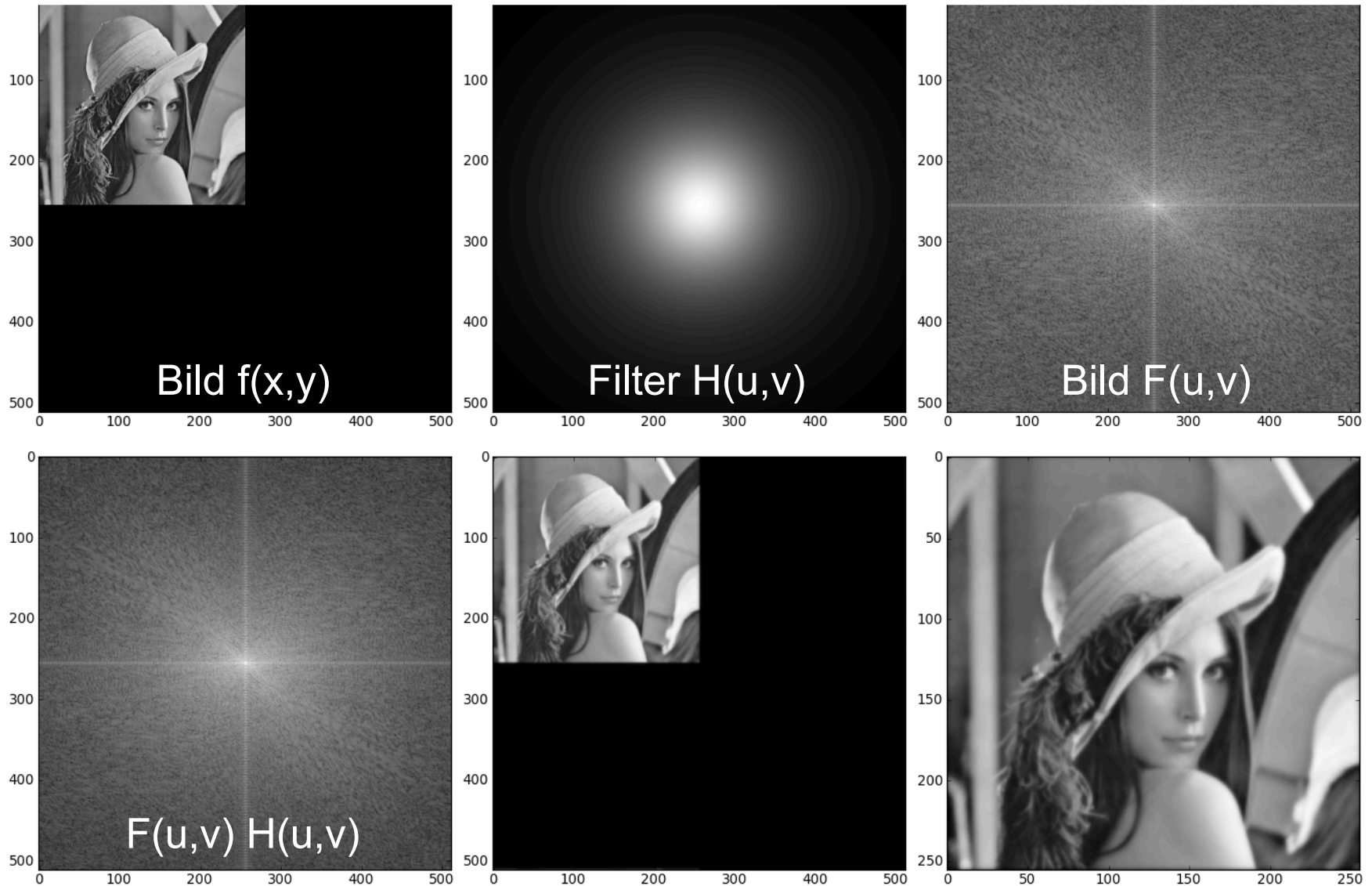
# Einfacher Tiefpass vs. Butterworth



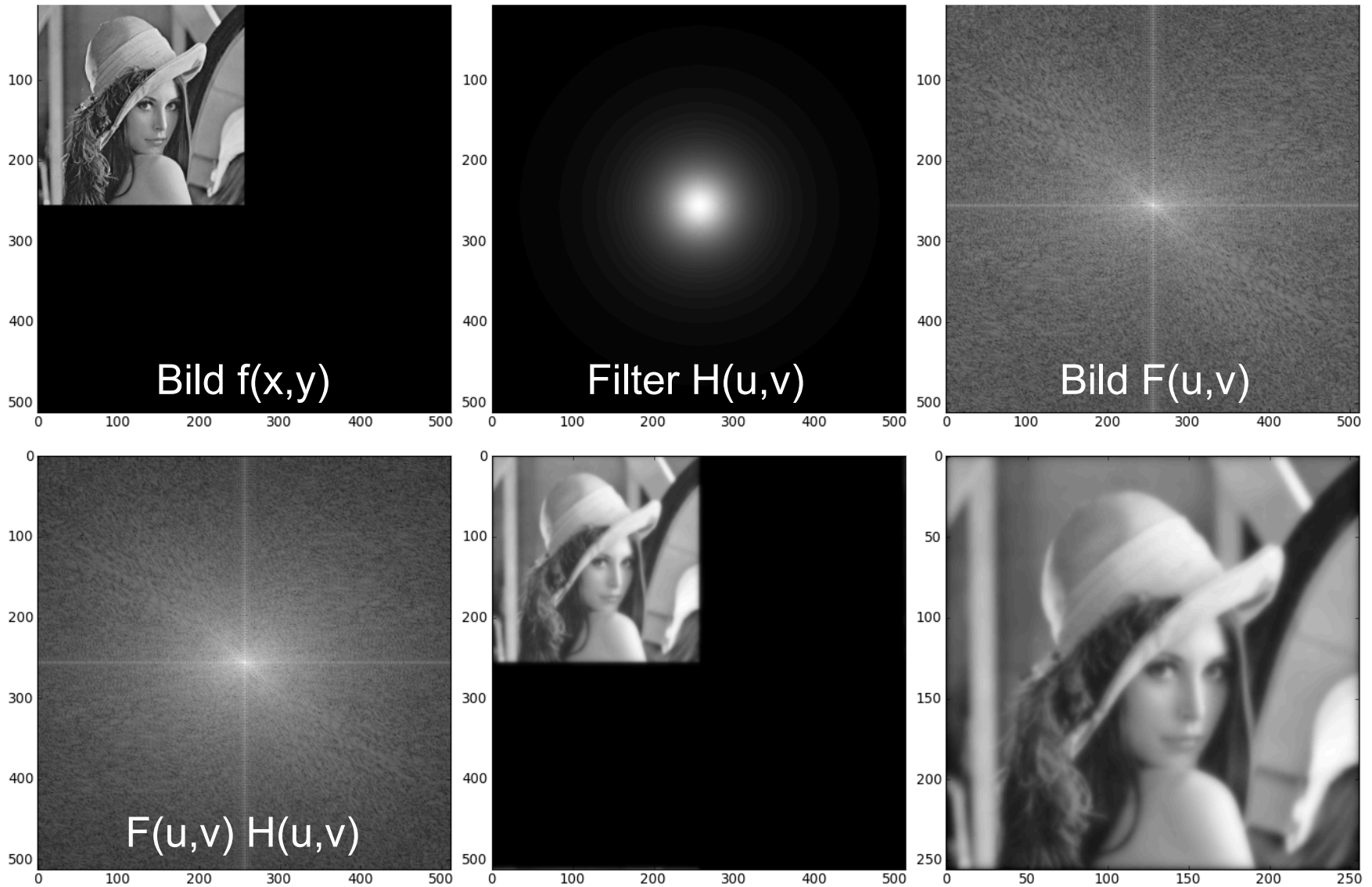
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# Beispiel: BLPF, $n = 2$ , $D_0 = 80$

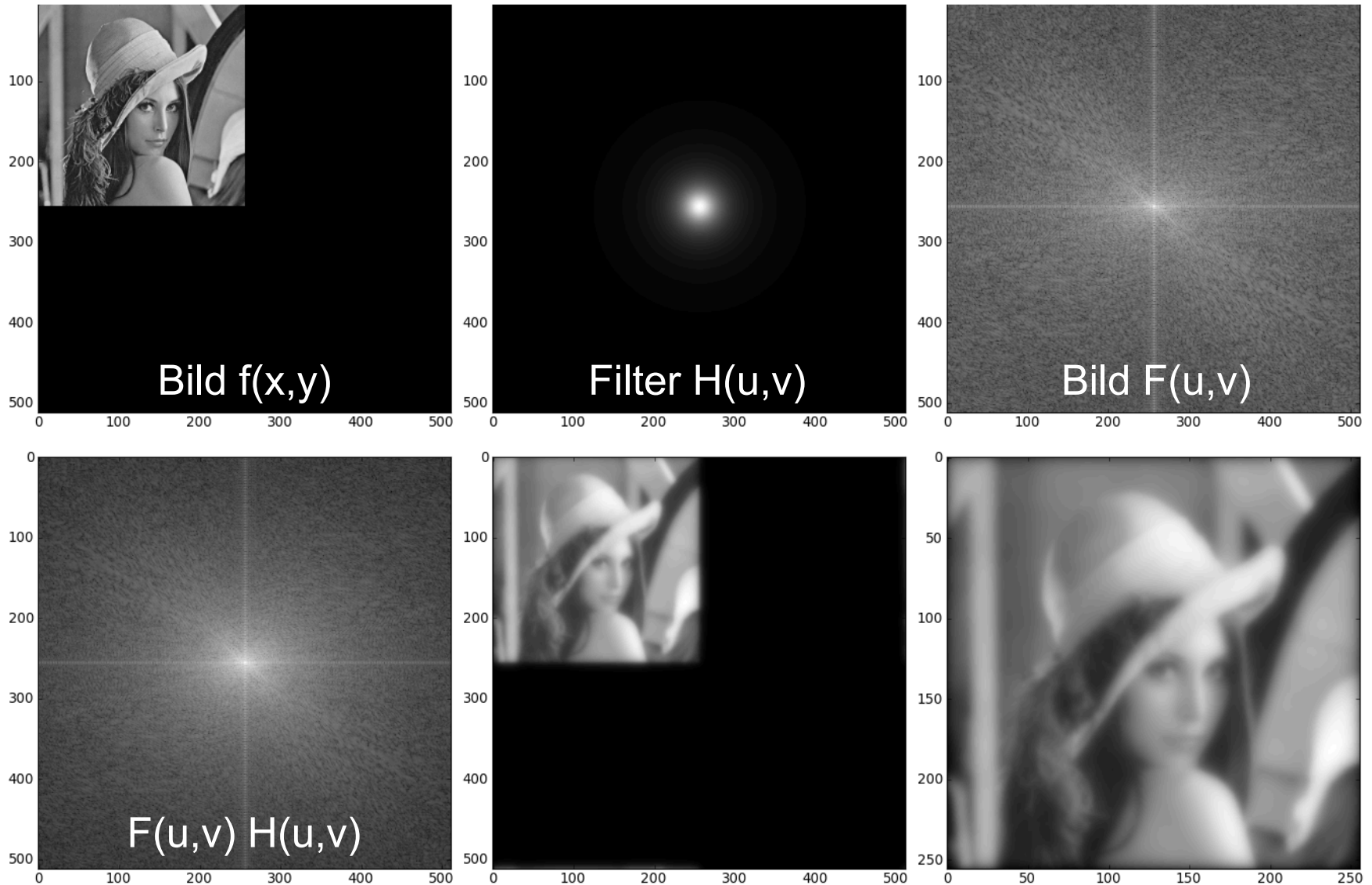


# Beispiel: BLPF, $n = 2$ , $D_0 = 40$



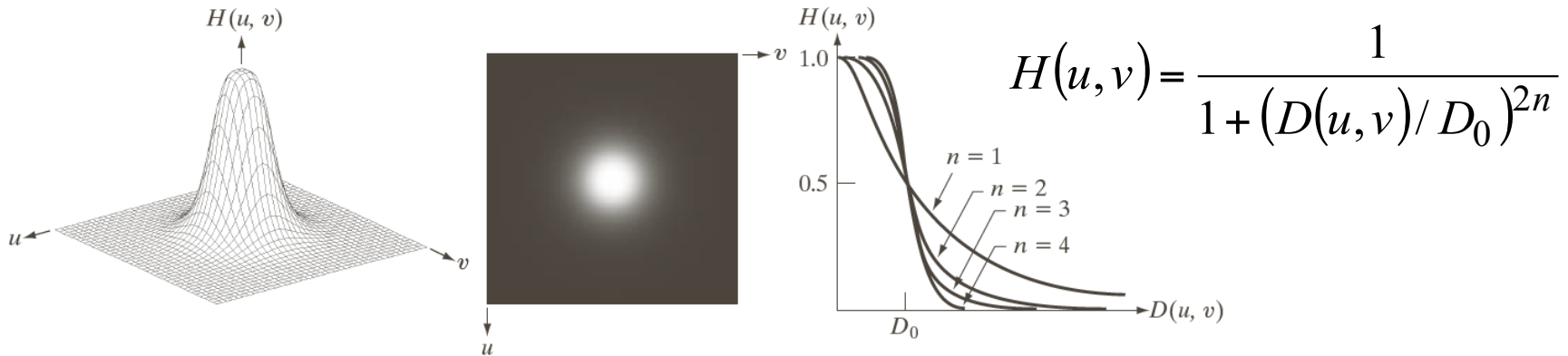


# Beispiel: BLPF, $n = 2$ , $D_0 = 20$



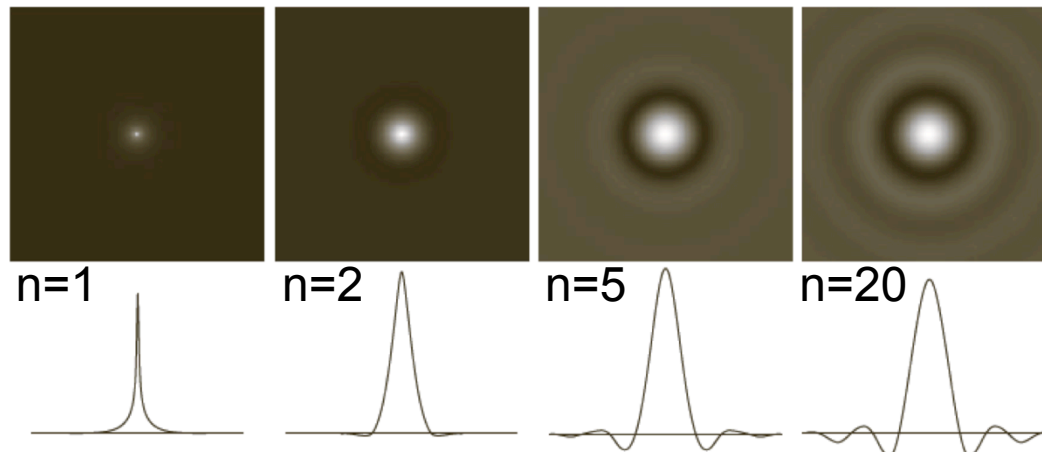
# Ordnung n des Butterworth-Filters

- Ordnung n bestimmt die Steilheit der Transferfunktion



- je größer Ordnung n, desto stärkere Ringing-Artefakte

$M=N=1000$   
 $D_0=5$



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# Gauß-Filter

- Keine Artefakte, da Fourier-Transformation einer Gauß-Funktion wieder eine Gauß-Funktion

- Tiefpass-Filter (GLPF):

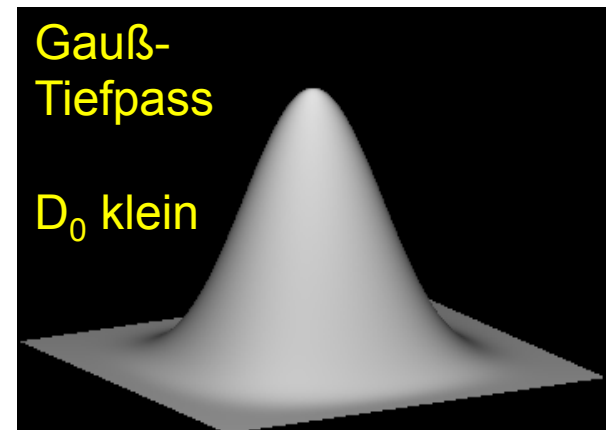
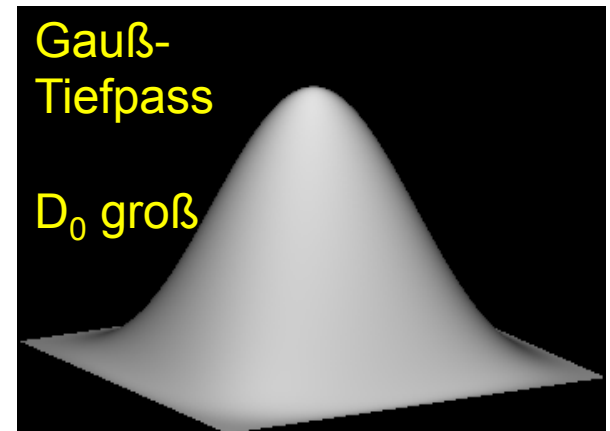
$$H(u, v) = \exp\left(-\frac{D^2(u, v)}{2D_0^2}\right)$$

- Hochpass-Filter (GHPF):

$$H(u, v) = 1 - \exp\left(-\frac{D^2(u, v)}{2D_0^2}\right)$$

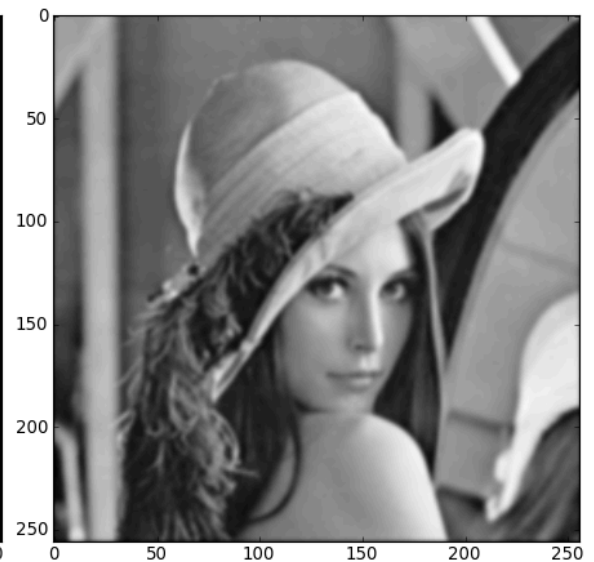
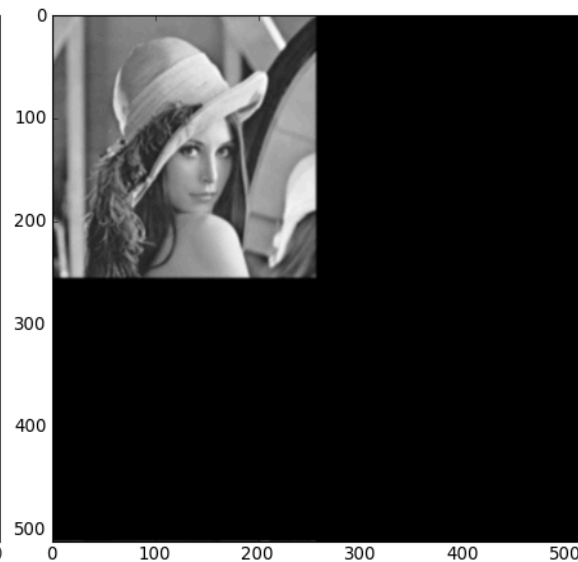
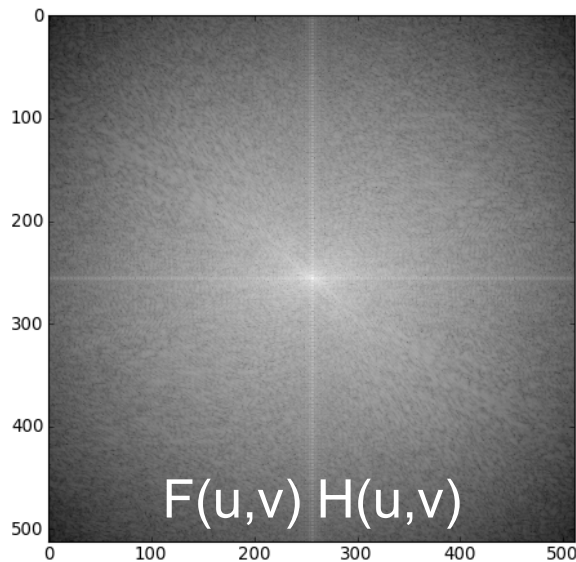
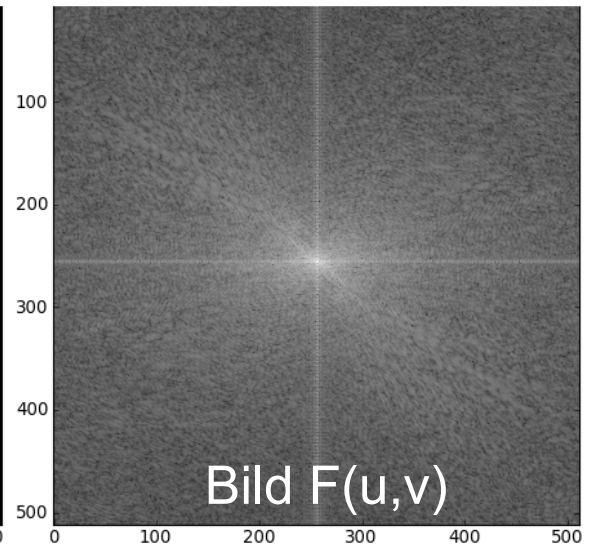
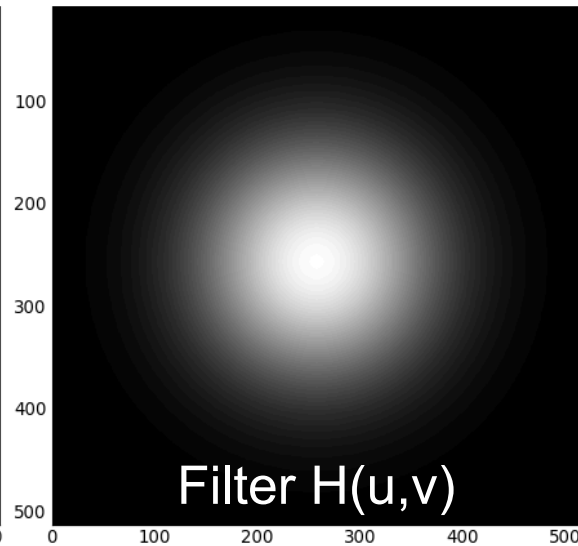
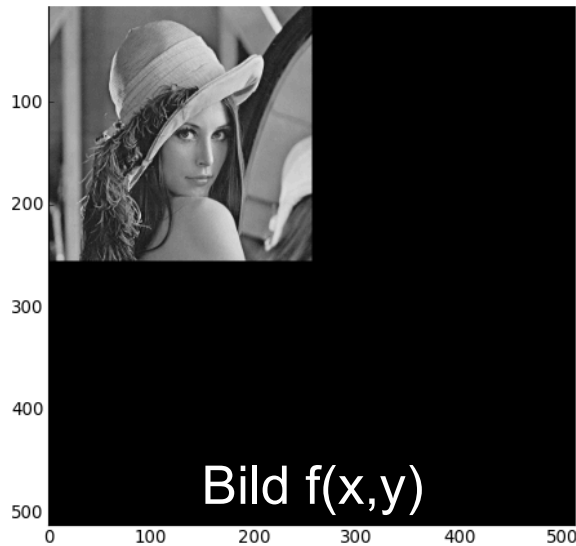
$D_0$ : entspricht  $\sigma$

$D(u, v)$ : Frequenz, Abstand vom Ursprung

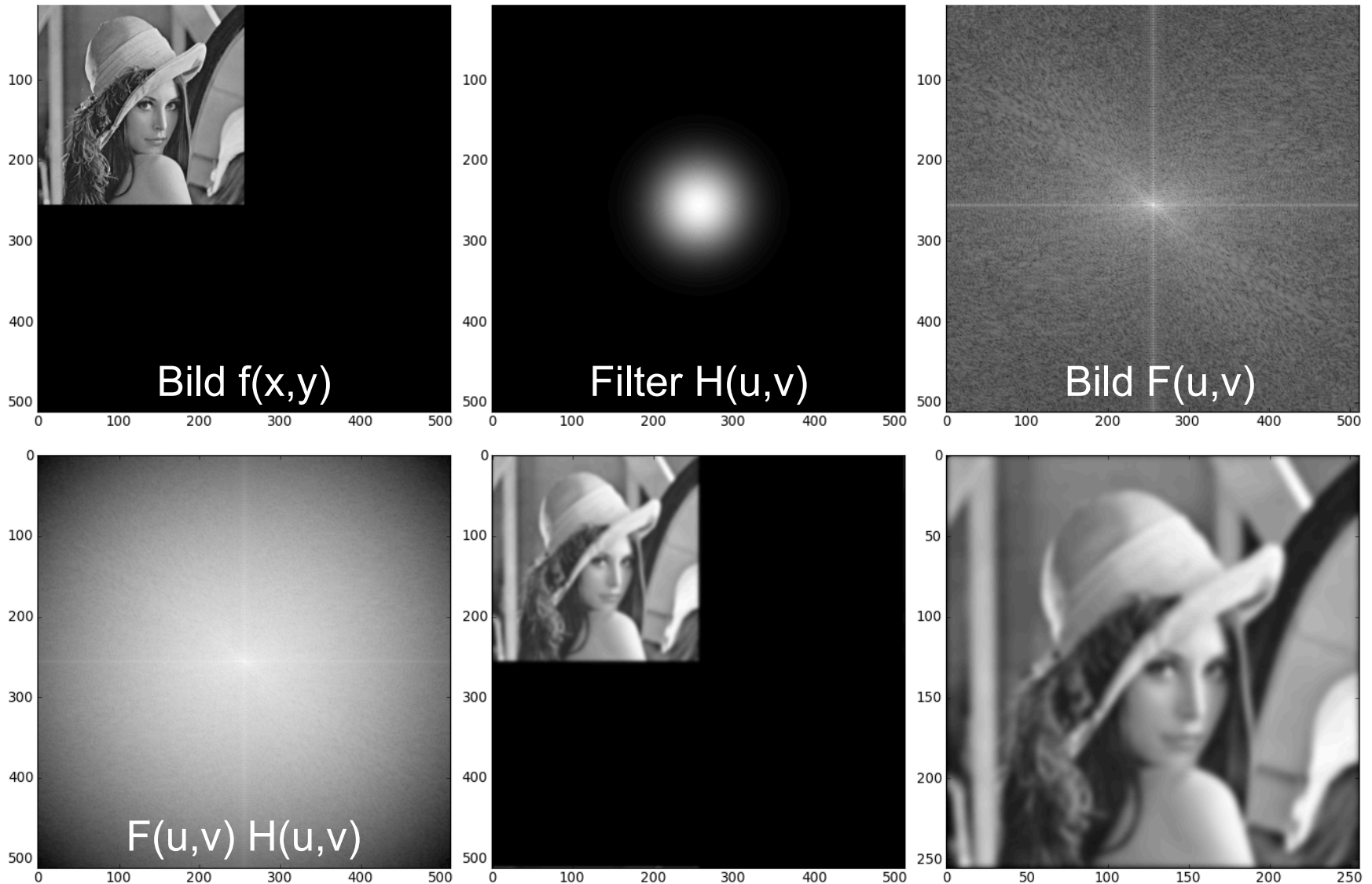


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# Beispiel: GLPF, $D_0 = 80$

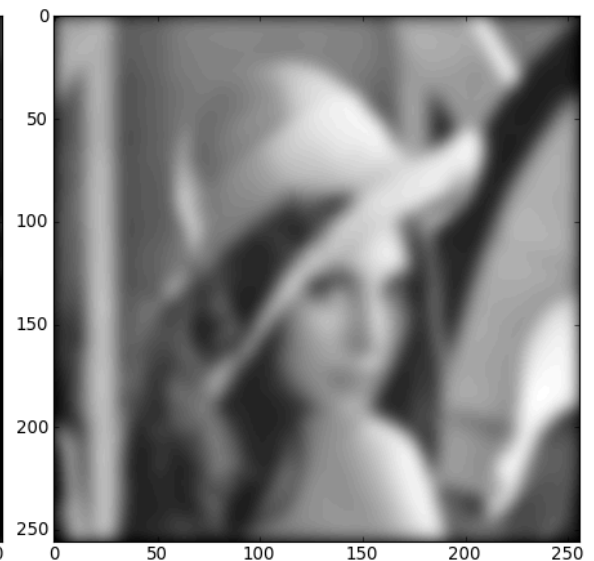
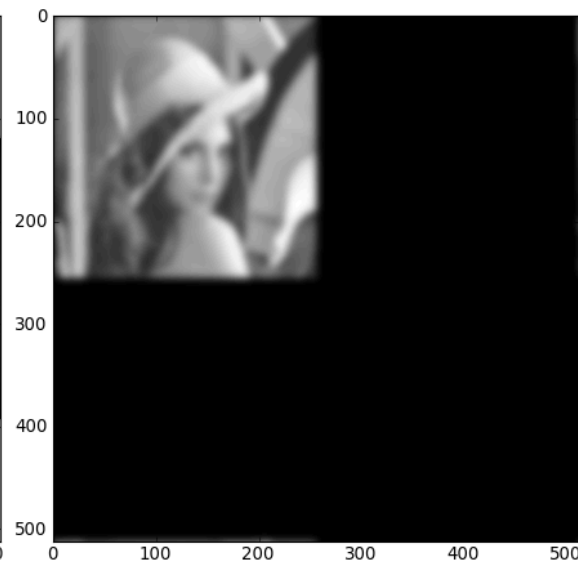
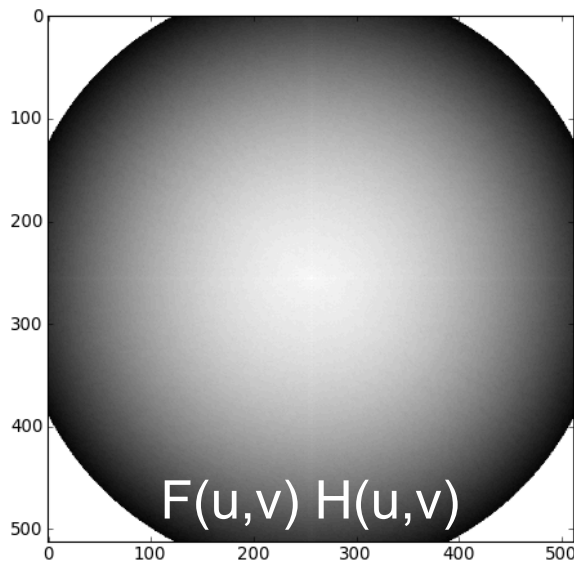
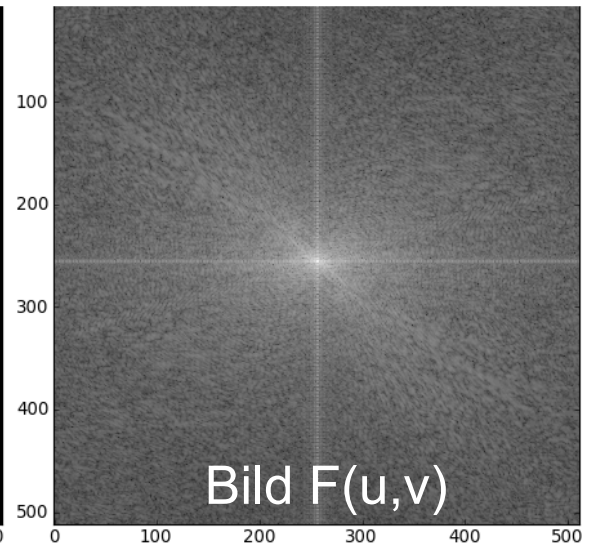
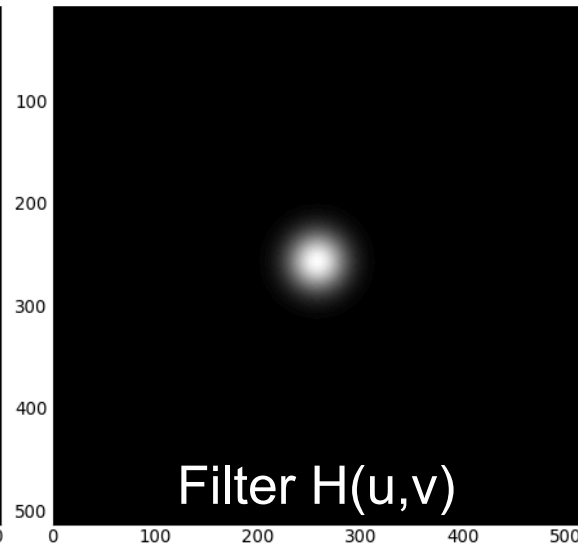
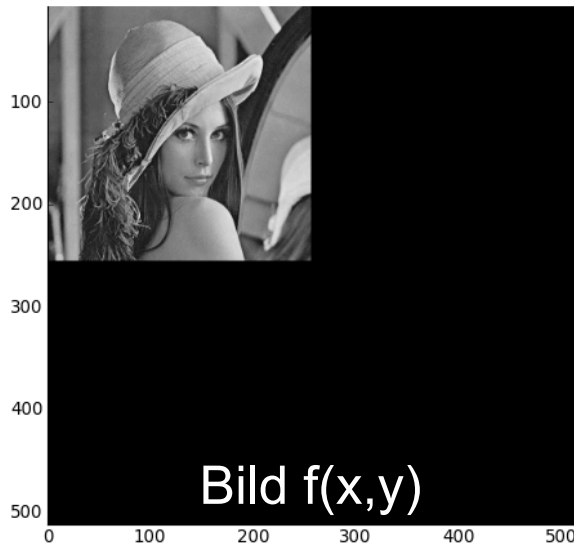


# Beispiel: GLPF, $D_0 = 40$



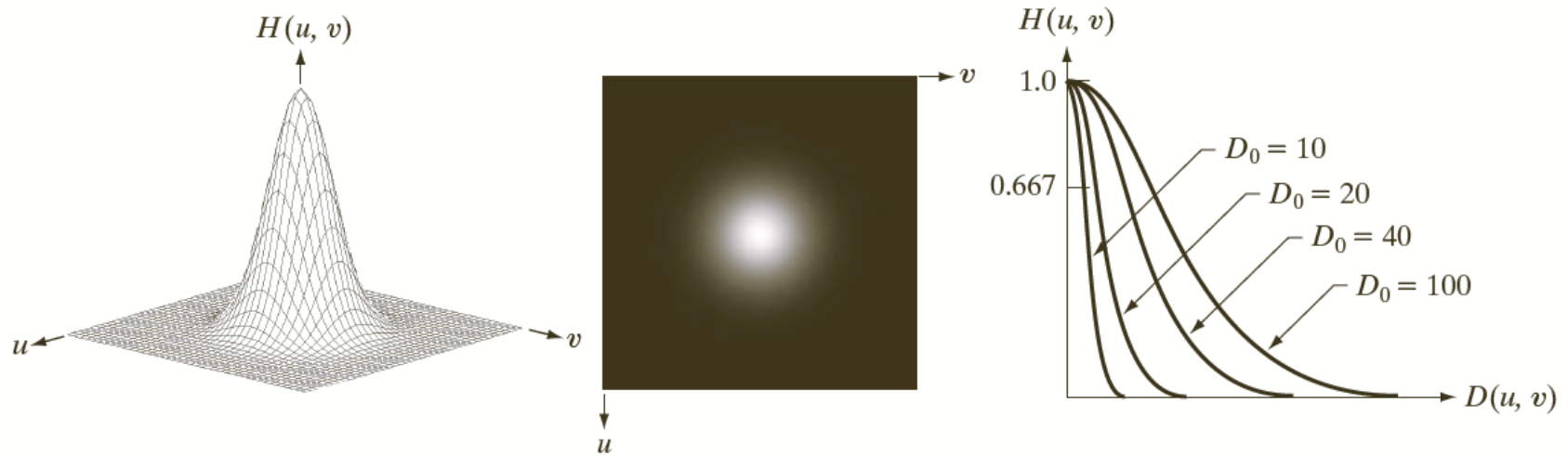


# Beispiel: GLPF, $D_0 = 20$



# Transferfunktion des GLPF

- GLPF begrenzt weniger scharf, als das BLPF
- GLPF erzeugt keine Ringing-Artefakte



a b c

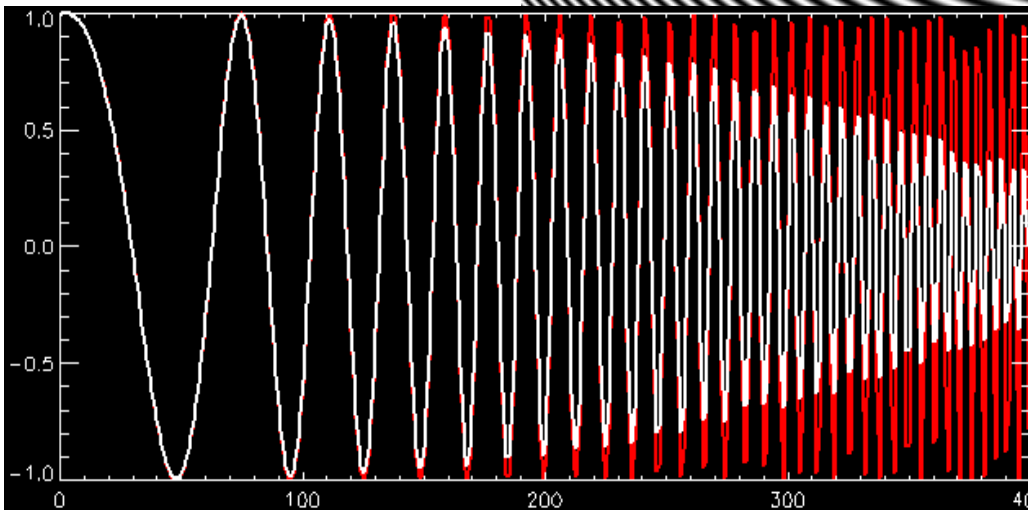
**FIGURE 4.47** (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of  $D_0$ .

# Filterresultate des Gauß-Filters

original

Filter B<sup>16</sup>

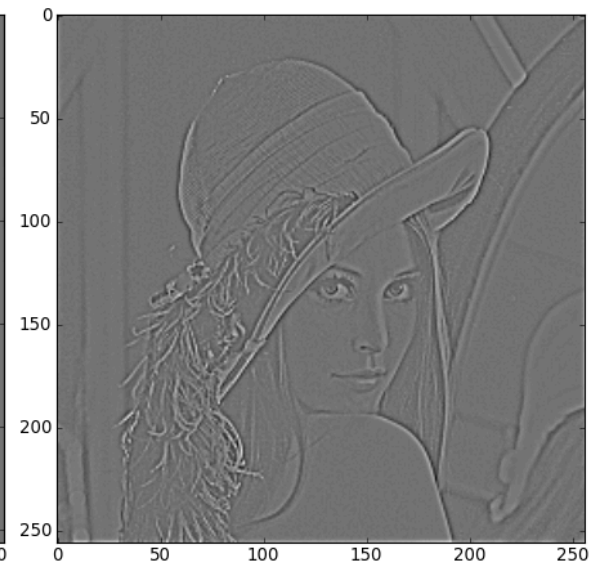
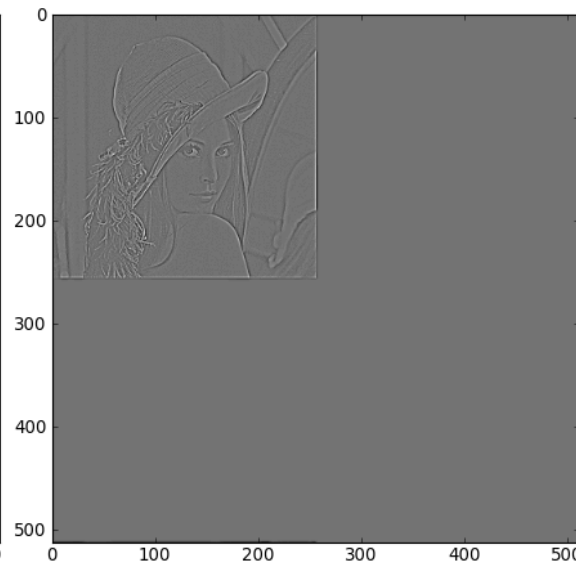
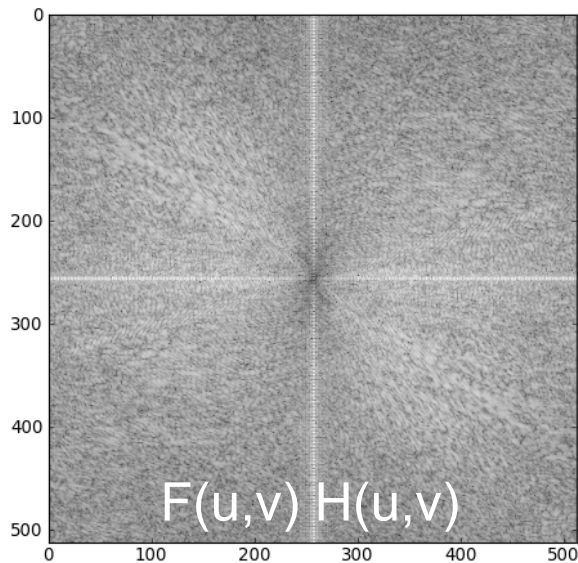
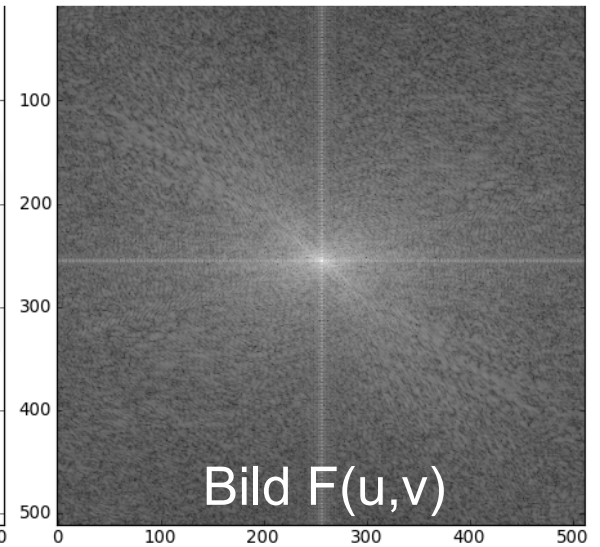
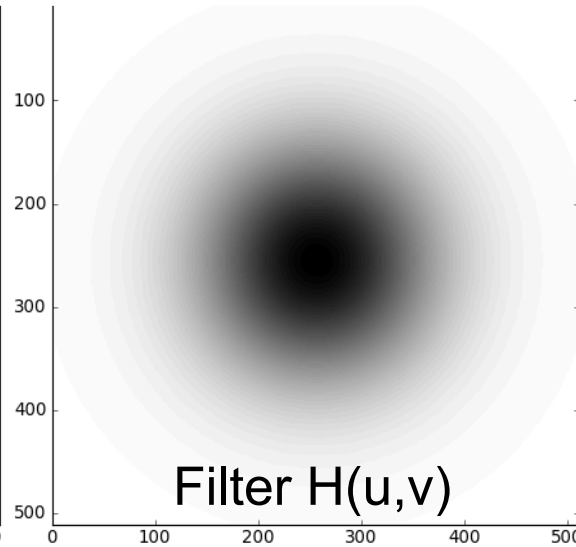
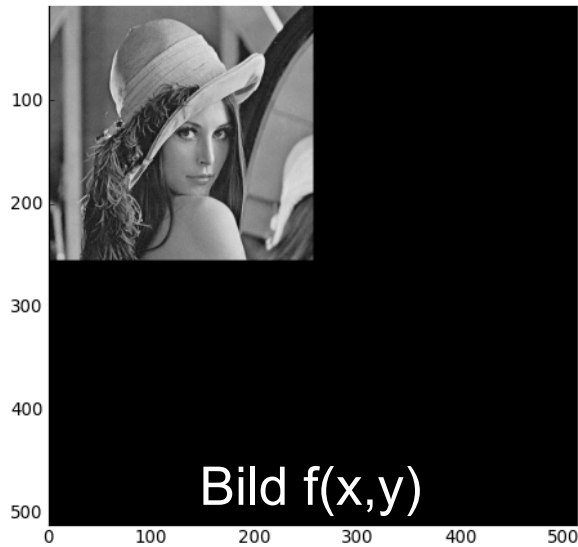
Bildzeile  
rot: vor der Filterung  
weiß: nach Filterung



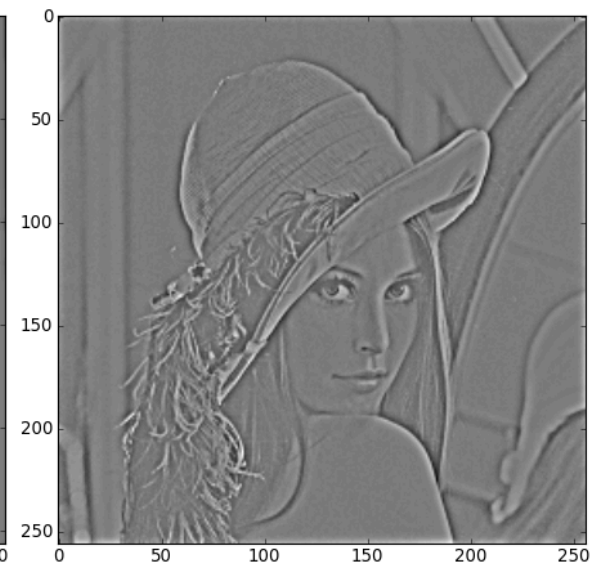
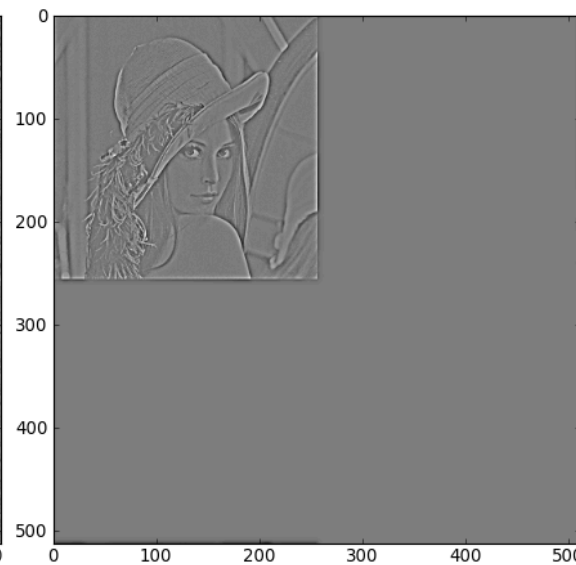
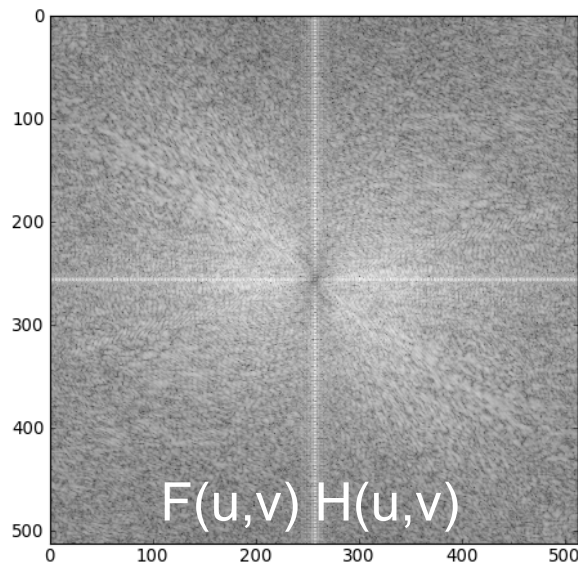
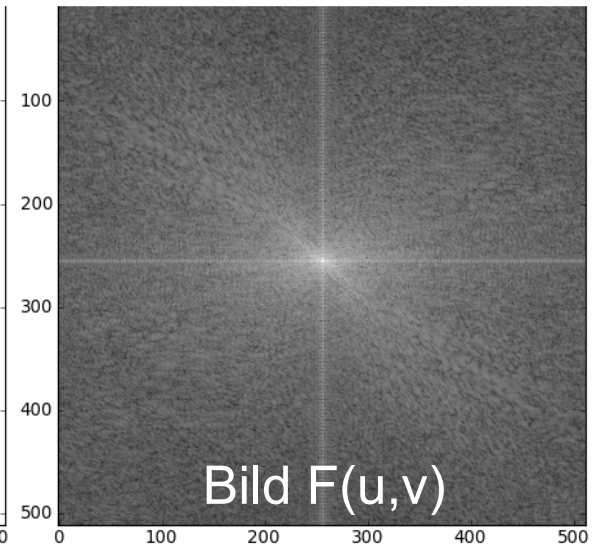
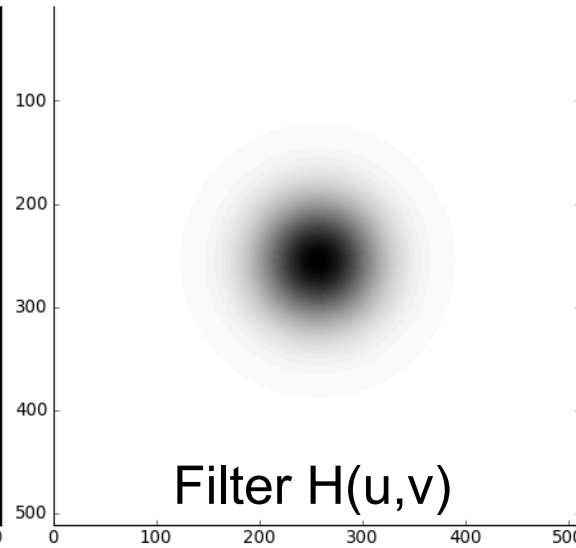
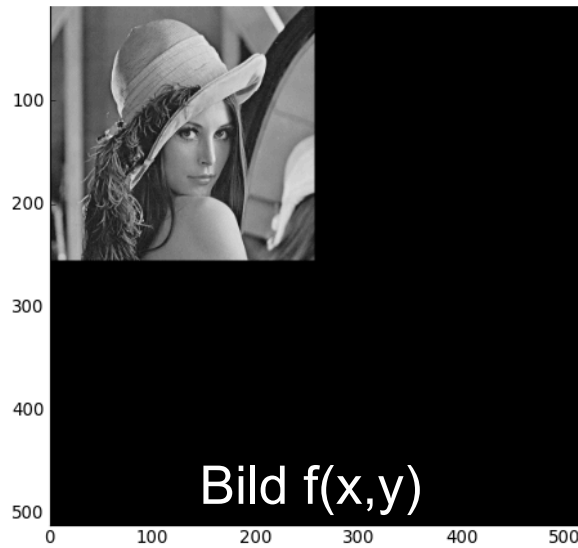
keine Artefakte  
erkennbar



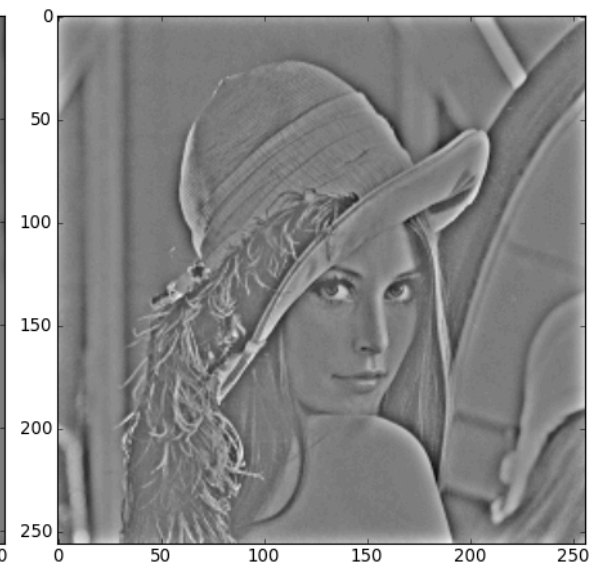
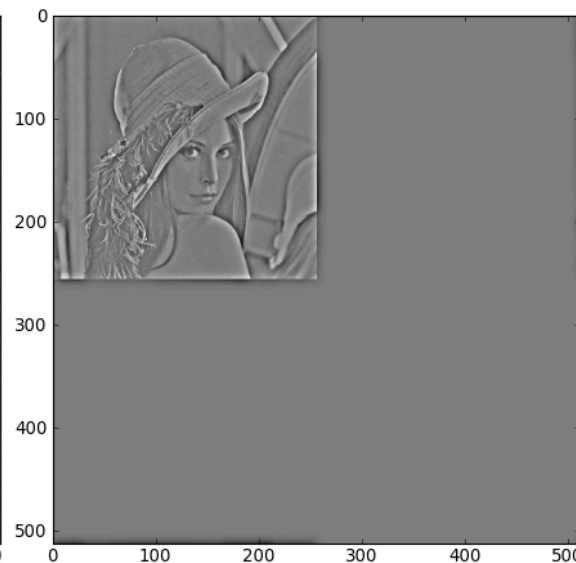
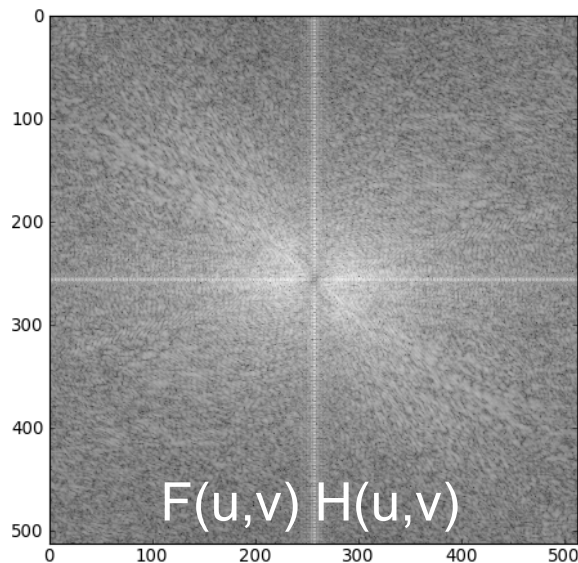
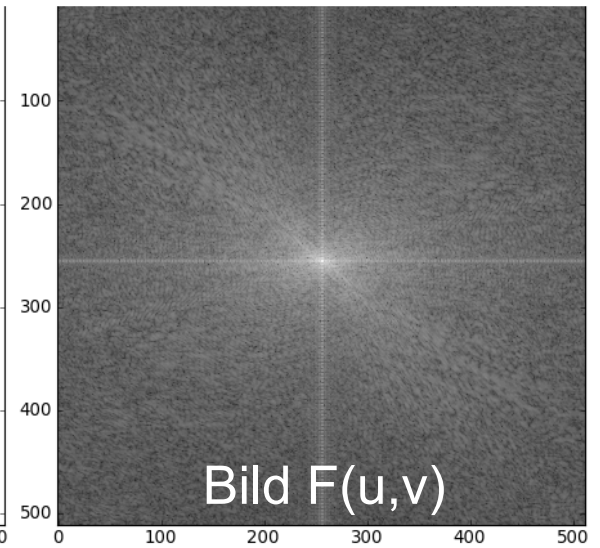
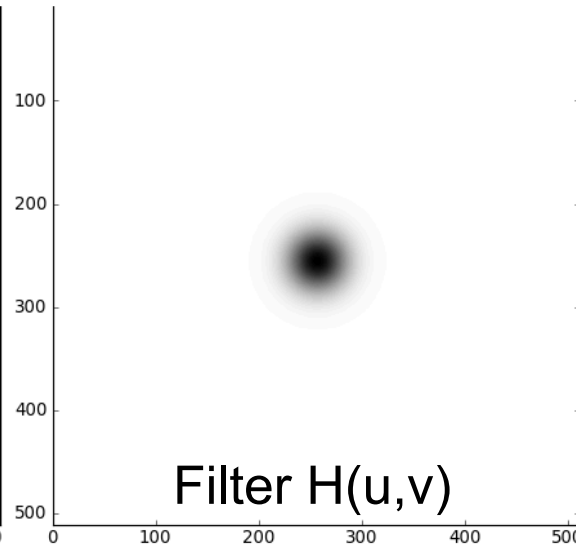
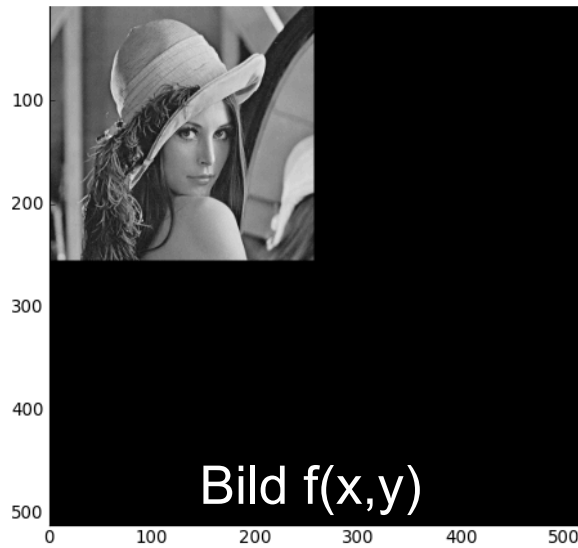
# Beispiel: GHPF, $D_0 = 80$



# Beispiel: GHPF, $D_0 = 40$



# Beispiel: GHPF, $D_0 = 20$





# Vermeiden der Auslöschung niedriger Frequenzen beim GHPF

- Hochpass-Filter (GHPF):

$$H(u, v) = 1 - \exp\left(-\frac{D^2(u, v)}{2D_0^2}\right)$$

$0 \leq H(u, v) \leq 1$

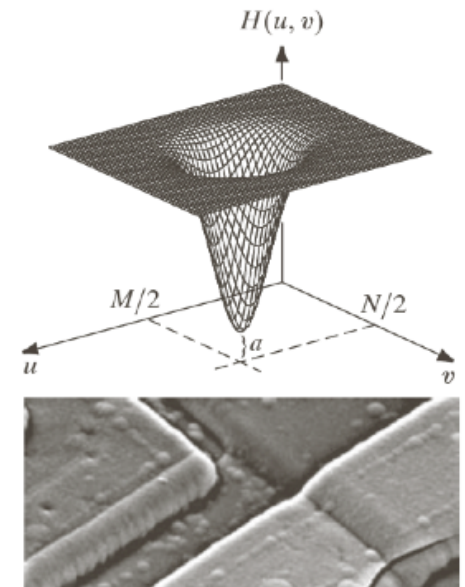
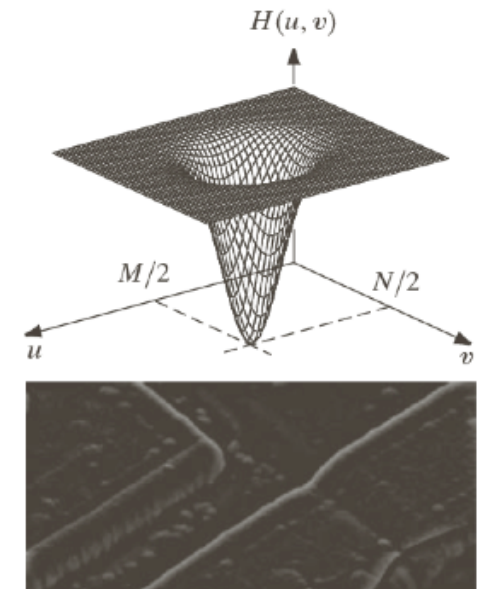
- modifiziertes GHPF:

$$H_a(u, v) = a + (1 - a) \cdot H(u, v)$$

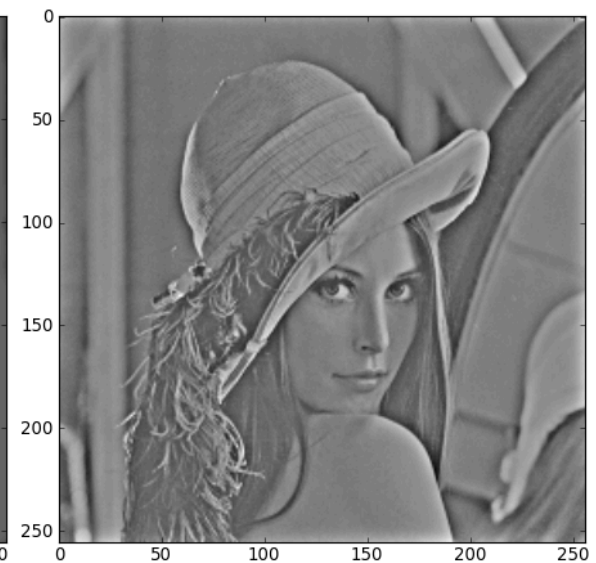
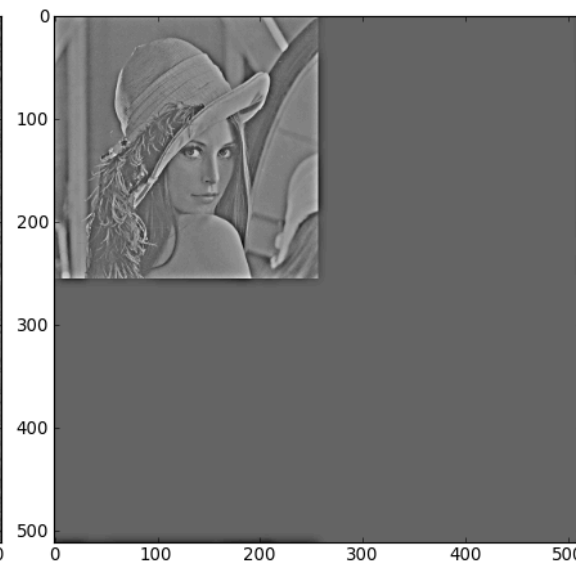
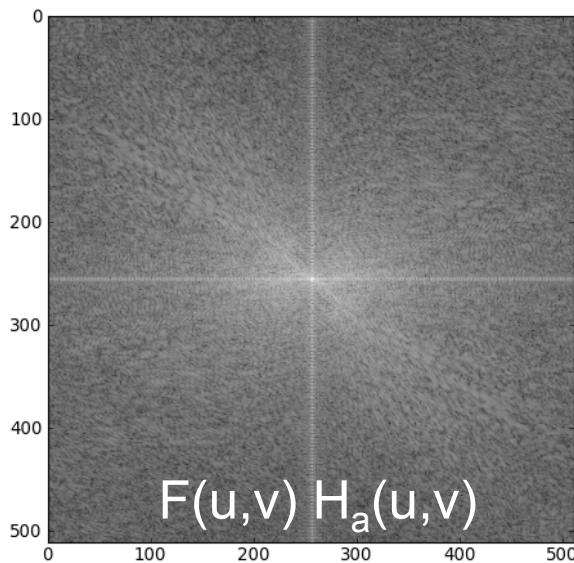
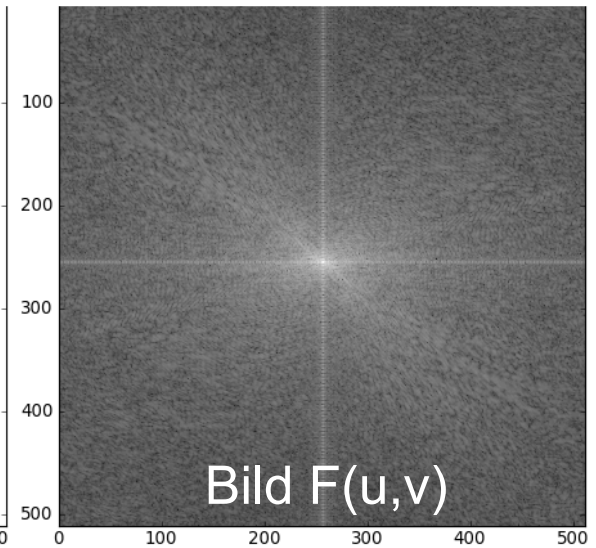
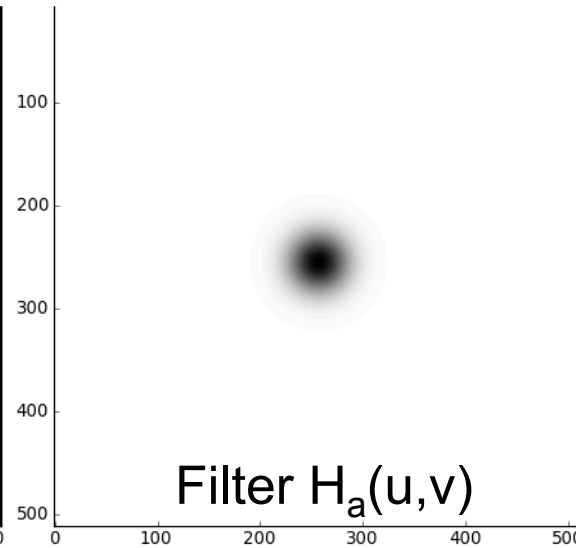
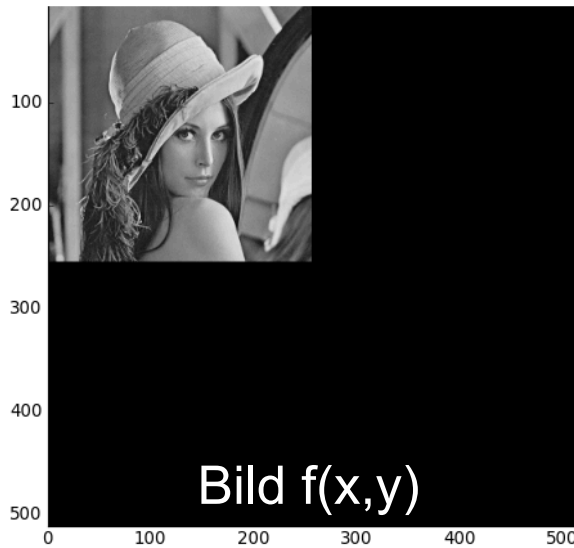
$a \leq H_a(u, v) \leq 1$

$D_0$ : entspricht  $\sigma$

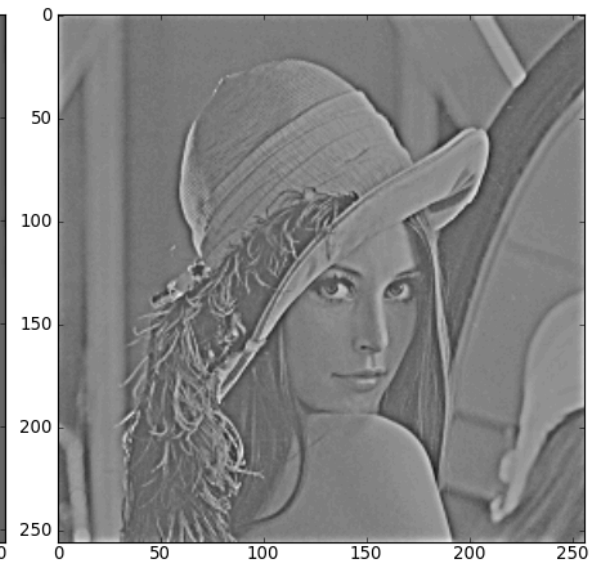
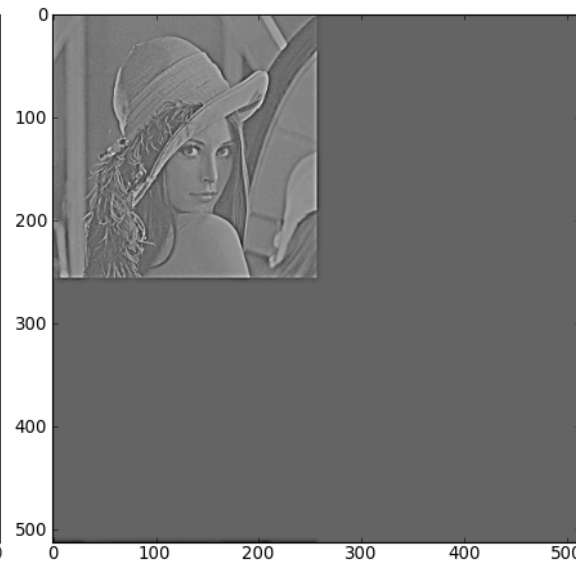
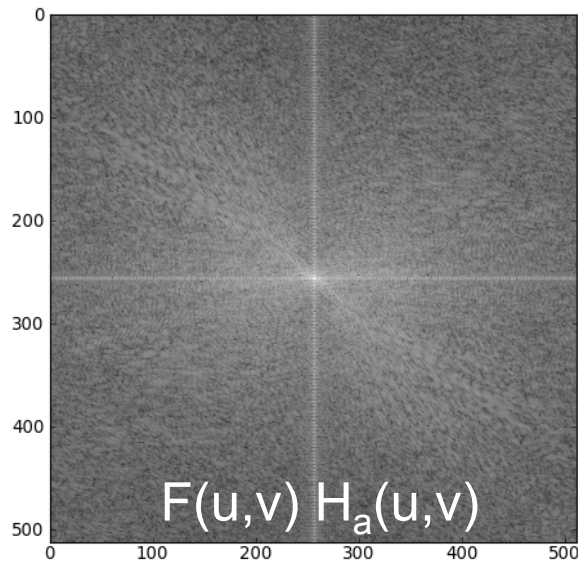
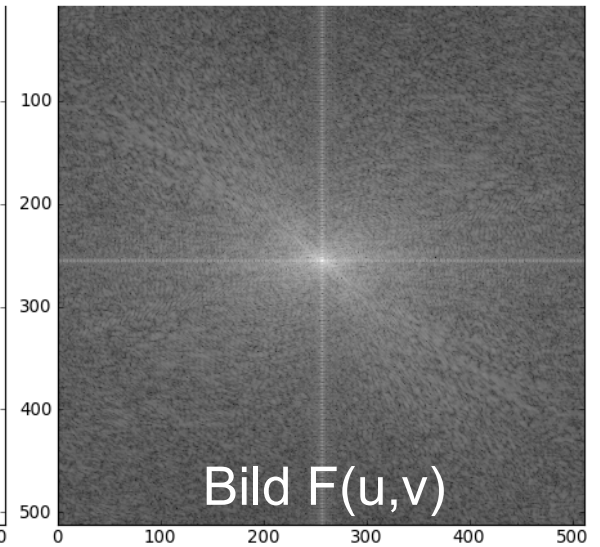
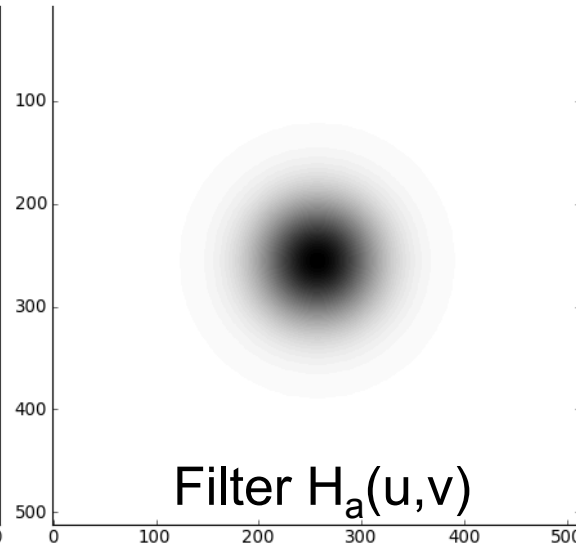
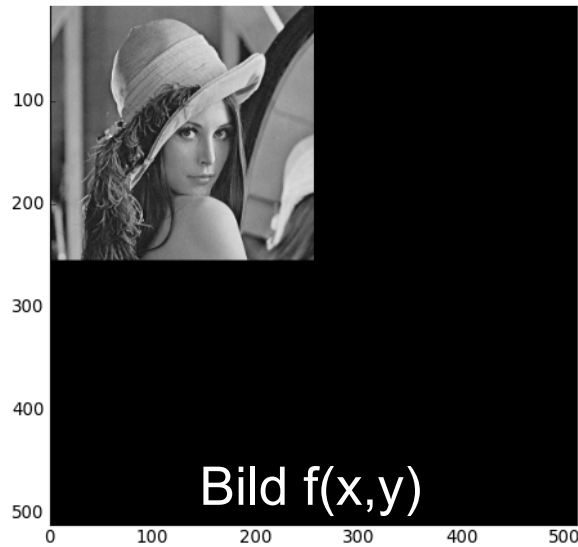
$D(u, v)$ : Frequenz, Abstand vom Ursprung



# Beispiel: GHPF, $D_0 = 20$ , $H_a(u,v)$ , $a = 0.2$



# Beispiel: GHPF, $D_0 = 40$ , $H_a(u,v)$ , $a = 0.2$



# Beispiel: GHPF, $D_0 = 80$ , $H_a(u,v)$ , $a = 0.2$

